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E–convex programming

J.Sheiba Grace , Research Scholar, Aditanar College, India, pthangavelu_2004@yahoo.co.in
P.Thangavelu, Professor, Aditanar College, India, ptvelu12@gmail.com

Abstract

Youness introduced the concepts of E–convex sets and E–convex functions and studied their properties. In this paper it is shown that Theorem 3.1 of Youness is incorrect by giving some counter examples. Theorem 3.1 of Youness is modified and the modified theorem is applied to solve an E–convex programming problem.

1. Introduction

The concepts of E–convex sets and E–convex functions introduced by Youness [7] have applications in mathematical sciences. Following this Xiusu Chen[6] introduced a new class of semi E–convex functions and discussed their basic properties. Also Suneja, Lalitha and Govil [4, 5] have studied E–convex functions and generalized E–convex functions and applied these functions to non linear programming problems. The authors [2, 3] studied some properties of E–convex sets and E–convex functions. In this paper it is shown that Theorem 3.1 of Youness [7] is incorrect by giving some counter examples. Throughout this paper R denotes the set of all real numbers. For the basic definitions the readers may consult [1]. Definitions and results that are required in sequel are listed below.

Definition 1.1: Let $E: R^n \rightarrow R^n$ be a map. A set $M \subseteq R^n$ is said to be E–convex in R^n if $\lambda E(x) + (1-\lambda)E(y) \in M$ for each $x, y \in M$, $0 \le \lambda \le 1$. [7]

Definition 1.2: Let $S \subseteq R^n \times R$ and $E: R^n \rightarrow R^n$. S is said to be E–convex in $R^n \times R$ if $(x, \alpha), (y, \beta) \in S$ imply $(\lambda E(x)+(1-\lambda)E(y), \lambda\alpha + (1-\lambda)\beta) \in S$, $0 \le \lambda \le 1$. [7]

Definition 1.3: Let $S \subseteq R^n \times R$ and $E: R^n \rightarrow R^n$. An E–epigraph of a function $f: R^n \rightarrow R$ is defined as E–epigraph of f $=\{(x, \alpha): x \in M, \alpha \in R, f(E(x)) \le \alpha\}$. [7]

Definition 1.4: Let $E: R^n \rightarrow R^n$ be a map. A function $f: R^n \rightarrow R$ is said to be E–convex on a set $M \subseteq R^n$ if M is an E–convex set and if $f(\lambda E(x)+(1-\lambda)E(y)) \le \lambda f(E(x))+(1-\lambda)f(E(y))$ for each $x, y \in M$, $0 \le \lambda \le 1$. [7]

Youness used Definition 1.1 to define E–convex functions and characterized E–convex functions using Definition 1.2 and Definition 1.3.

Theorem 1.5: A numerical function f defined on an E–convex set $M \subseteq R^n$ is E–convex on M if and only if E–epigraph of f is E–convex in $R^n \times R$. [7]

2. Counter examples

In the above theorem, the sufficient part namely "If the E–epigraph of a numerical function f defined on an E–convex set $M \subseteq R^n$ is E–convex in $R^n \times R$ then f is E–convex on M" is true. However the necessary part namely "If a numerical function f defined on an E–convex set $M \subseteq R^n$ is E–convex on M then the E–epigraph of f is E–convex in $R^n \times R$" is not true as shown in the following examples.

Example 2.1: Let f :R→ R be defined as f(x) = $\begin{cases} 1 \text{ if } x > 0 \\ -x \text{ if } x \le 0 \end{cases}$ and let

E:R→ R be defined as E(x) = - x^2. Then R is an E–convex set and f is E–convex on R. Then the E–epigraph = {(x, α): x ∈R, α∈R, f(E(x)) ≤ α }
= {(x, α): x∈R, α∈R, f($-x^2$) ≤ α }
= { (x, α): x∈R, α∈R, x^2 ≤ α }.

Clearly (1, 2), (-3, 9)∈E–epigraph of f. Then for $\lambda = \frac{1}{2}$, we see that

(λE(1)+(1-λ)E(-3), 2λ+9(1-λ))∉E–epigraph of f. Now λE(1)+(1-λ)E(-3) = -5 and 2λ + 9(1-λ) = 5.5. As $(-5)^2$ = 25 > 5.5, (-5, 5.5)∉E–epigraph of f. Therefore the E–epigraph of f is not E–convex in R×R.

Example 2.2: Let f:R^2 →R be defined as f(x ,y) = x-2. Let E:R^2 →R^2 be defined as E(x, y) = (y, x). Then R^2 is an E–convex set and f is E–convex on R^2.
E–epigraph of f = {(x, y, α) : (x, y)∈R^2, α∈R, f(E(x, y)) ≤ α }
= {(x, y, α) : (x, y)∈R^2, α∈R, f((y, x)) ≤ α }
= {(x, y, α) : (x, y)∈R^2, α∈R, y-2 ≤ α }
= {(x, y, α) : (x, y)∈R^2, α∈R, y ≤ α + 2 } ……………(1)
Consider (4, -1, 0), (2, 2, 0)∈E–epigraph of f. Then
λE(4, -1)+(1-λ)E(2, 2) = λ(-1, 4)+(1-λ)(2, 2) = (-3λ +2, 2λ+2) and 0λ+0(1-λ)= 0. For λ =1,
(-3λ +2, 2λ+2) = (-1, 4). As 4 >0+2, by (1), (-1, 4, 0)∉E–epigraph of f. Therefore E-epigraph of f is not E–convex in R^2× R.

3. Rectification

In this section the necessary part of Theorem 1.5 is modified and its validity is established.

Theorem 3.1: Let M⊆R^n be an E–convex set. Suppose E:R^n →R^n is linear and idempotent. Let a numerical function f defined on M be E–convex on M. Then the E–epigraph of f is E–convex in R^n× R.

Proof: Let (x, α), (y, β)∈E–epigraph of f. Then f (E(x))≤ α ; f(E(y)) ≤ β.
f(E(λE(x) + (1-λ)E(y))) = f(λE^2(x) + (1-λ)E^2(y))
= f(λE(x) + (1-λ)E(y))
≤ λf(E(x)) + (1 -λ)f(E(y))
≤ λα + (1-λ)β.
Then using Definition 1.3, (λE(x) + (1-λ)E(y), λα + (1-λ)β)∈E–epigraph of f. This proves that the E–epigraph of f is E–convex . This completes the proof.

The following examples show that the condition that E is linear / idempotent cannot be dropped.

Example 3.2: Let f:R→R be defined as f(x) **=** $\begin{cases} 1 \text{ if } x > 0 \\ -x \text{ if } x \le 0 \end{cases}$.

Let E:R→R be defined as E(x) = - x^2. Then R is an E–convex set and f is E–convex on R. But the E–epigraph of f is not E–convex in R× R. It is easy to see that E is neither linear nor idempotent.

Example 3.3: Let f :R→R be defined as f(x) = | x |. Let E:R→R be defined as E(x) = kx where k>0. Then R is an E–convex set and f is E–convex on R.
E–epigraph of f = {(x, α) **:** x∈R, α ∈ R, f (E(x)) ≤ α }

$= \{(x, \alpha) : x \in R, \alpha \in R, f(kx) \leq \alpha\}$
$= \{(x, \alpha) : x \in R, \alpha \in R, |kx| \leq \alpha\}$
$= \{(x, \alpha) : x \in R, \alpha \in R, k|x| \leq \alpha\}$
$= \{(x, \alpha) : x \in R, \alpha \in R, |x| \leq \frac{\alpha}{k}\}$.

Now fix k = 2. Then (- 1, 2), (0, 0)$\in$E–epigraph of f. For $\lambda = \frac{1}{2}$,

$\lambda E(-1)+(1-\lambda)E(0) = \frac{1}{2}(-k) + \frac{1}{2}(0) = \frac{1}{2}(-2) = -1$. $\lambda(2) + (1-\lambda)0 = \frac{1}{2}(2) = 1$.

As $|-1| > \frac{1}{2}$, $(-1, 1) \notin$ E–epigraph of f. Therefore the E–epigraph of f is not E–convex. Here E is linear but not idempotent .

4. Applications to E-convex programming

Youness [7] considered the following E–convex programming problem.
(EP): Minimize f(x), $x \in M = \{x \in R^n : g_i(x) \leq 0, i=1,2,\ldots m\}$ where
$f:R^n \to R$ and $g_i: R^n \to R$, i=1,2,…m are E–convex functions on R^n .
Youness proved certain results of (EP) in [7] namely Result 4.2, Result 4.3 and Result 4.4. But Xiusu Chen [6] has given some counter examples for Result 4.2, Result 4.3 and Result 4.4. The example given below is a counter example for Result 4.1. Youness also proved (Result 4.1) that the set M associated with the E–convex programming (EP) is E–convex. But this is not true as shown in the Example 4.5.

Result 4.1 (Theorem 4.1, [7]): The set M is E–convex.

Result 4.2 (Theorem 4.2, [7]): Assume that E(M) is convex and x* is a solution of the problem (EP_E): min(f oE)(x), $x \in M$. Then, E(x*) is a solution of problem (EP).

Result 4.3 ((Theorem 4.3, [7]): Let E(M) be a convex set. If $x^* = E(z^*) \in E(M)$ is a local minimum of Problem (EP) on M, then x* is a global minimum of Problem (EP) on M.

Result 4.4 ((Theorem 4.6, [7]): The set of all optimal solutions of Problem (EP) is convex.

Example 4.5: Let $g_i:R \to R$, i=1,2 and $f:R \to R$ be defined as $g_1(x)=3x$, $g_2(x)=-|x|$ and
$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}$. Define E: R→ R as E(x) = | x |. Then f, g_1 and g_2 are E–convex functions on R. Consider the E–convex programming problem
(EP): Minimize f (x), $x \in M = \{x \in R : g_1(x) \leq 0, g_2(x) \leq 0\}$.

Then M is not E–convex. Since $g_1(-3) = -9$ and $g_2(-3) = -3$, $-3 \in M$.
Also since $g_1(0) = 0 = g_2(0)$, $0 \in M$. Let $0 \leq \lambda \leq 1$. Then $\lambda E(-3)+(1-\lambda)E(0) = 3\lambda$. Since $g_1(3\lambda) = 9\lambda > 0$, $3\lambda \notin M$ that implies M is not E–convex.
In the above example, E is not linear. This has a motivation to modify the E–convex programming problem.

(EQ): Minimize f(x), $x \in M = \{x \in R^n : g_i(E(x)) \leq 0, i=1,2,\ldots m\}$ where
(a) $f:R^n \to R$ and $g_i: R^n \to R$, i=1,2,…m are E–convex functions on R^n
(b) the operator $E: R^n \to R^n$ is linear and idempotent.

Theorem 4.6: The set M is E–convex.

Proof: Since g_i , i =1,2,…,m, are E–convex functions and since E is linear and idempotent, using Theorem 3.1, the E–epigraph of each g_i is E–convex. Suppose x, y ∈M and $0 \le \lambda \le 1$. Then from (EQ), $g_i(E(x)) \le 0$ and $g_i(E(y)) \le 0$. Using Definition 1.3, (x, 0), (y, 0)∈E–epigraph of g_i.
Since the E–epigraph is E–convex ($\lambda E(x)+(1-\lambda)E(y)$, 0)∈E–epigraph of g_i. Then again by Definition 1.3, $g_i(E(\lambda E(x)+(1-\lambda)E(y))) \le 0$. From **(EQ)** it follows that M is E–convex.

Theorem 4.7: Assume that E(M) =M and x* is a solution of the problem
(EQ_E): min (foE)(x), x∈M. Then, E(x*) is a solution of the problem **(EQ)**.

Proof: Suppose E(x*) is not a solution of (EQ). Then, there is x′∈M such that
f(x′) < f(E(x*)). Since M = E(M), there is y∈M such that x′ =E(y) and
f(x′) = f(E(y))< f(E(x*)) which contradicts the optimality of x* for Problem **(EQ_E)**. Hence E(x*) is a solution for Problem **(EQ)**.

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