

The Relationships among Number Theoretic Functions

Mohammed R. Karim, Associate Professor, Alabama A&M University, mohammed.karim@aamu.edu

1. Abstract

There are several number theoretic functions which play important roles in multiplicative and additive number theory. Earlier authors have established relationships among some of these functions, but not all of them. In this paper, the author presents relationships between some of the remaining functions. The paper shows relationships among most of the number theoretic functions, making it possible to find the relationship between any two number theoretic functions.

2. Introduction

The goal of this paper is to show the relationships among different number theoretic functions. In order to do that, one needs to describe all the number theoretic functions concerned with this paper along with their properties. First, the author defined the respective functions, presented simple examples and then stated some theorems related to each function. These theorems are standard results in Number Theory and their proofs are found in any standard book of Number Theory [1, 2]. Thus, he did not provide any proof of these theorems. He will prove the results he wanted to establish.

3. Number Theoretic Functions

Definition 1: A function f is said to be a *number theoretic function*, also known as an *arithmetic function*, if its domain is the set of all positive integers.

The present work will involve the following number theoretic functions

1. $\tau(n)$, the number of positive divisors of n
2. $\sigma(n)$, the sum of the positive divisors of n
3. $\phi(n)$, the Euler Phi function
4. $\mu(n)$, the Möbius function
5. $p(n)$, the partition function

The description each of these functions along with some of their properties are provided below.

The Number of Positive Divisors Function

Definition 2: Let n be a positive integer. The *number of positive divisors function*, denoted by $\tau(n)$, is the function defined by

$$\tau(n) = |\{d \in \mathbb{Z} : d > 0; d \mid n\}|,$$

where \mathbb{Z} is the set of integers. That is, $\tau(n)$ is the number of positive divisors of n .

For example, let $n=20$. Then the positive divisors of n are 1, 2, 5, 4, 10, and 20 and $\tau(20) = 6$.

Theorem 1: If p is a prime number and $a \in \mathbb{Z}$ with $a \geq 0$, then $\tau(p^a) = a + 1$.

Theorem 2: If $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ with p_1, p_2, \dots, p_r distinct prime numbers and a_1, a_2, \dots, a_r are nonnegative integers, then

$$\tau(n) = \prod_{i=1}^r (a_i + 1).$$

The Sum of Positive Divisors Function

Definition 3: If $n \in \mathbb{Z}$ with $n > 0$, the *sum of positive divisors function*, denoted by $\sigma(n)$, is defined by

$$\sigma(n) = \sum_{d|n, d>0} d.$$

That is, $\sigma(n)$ is the sum of the positive divisors of n .

E.g. $\sigma(12) = 28$, since 1, 2, 3, 4, 6, and 12 are the divisors of 12 and the sum of them is $(1 + 2 + 3 + 4 + 6 + 12) = 28$.

Theorem 3: If p is a prime number and if $a \in \mathbb{Z}$ with $a \geq 0$, then

$$\sigma(p^a) = \frac{p^{a+1} - 1}{p - 1}.$$

Theorem 4: If $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ with p_1, p_2, \dots, p_r distinct prime numbers and a_1, a_2, \dots, a_r are nonnegative integers, then

$$\sigma(n) = \prod_{i=1}^r \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

The Euler Phi-Function

Definition 4: If $n \in \mathbb{Z}$ with $n > 0$, the *Euler phi-function*, denoted by $\phi(n)$, is the function defined by

$$\phi(n) = |\{x \in \mathbb{Z} : 1 \leq x \leq n; (x, n) = 1\}|.$$

That is, $\phi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n .

E.g. $\phi(14) = 6$, since there are six positive integers less than or equal to 14 that are relatively prime to 14, namely, 1, 3, 5, 9, 11, and 13.

Theorem 5: If p is a prime number and $a \in \mathbb{Z}$ with $a > 0$, then

$$\phi(p^a) = p^a - p^{a-1} = p^{a-1}(p - 1).$$

Theorem 6: If $n = \prod_{i=1}^r p_i^{a_i}$, then

$$\phi(n) = n \cdot \prod_{i=1}^r \left(\frac{p_i - 1}{p_i} \right).$$

The Möbius Function

Definition 5: For any positive integer n , the Möbius function, $\mu(n)$, is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^r, & \text{if } n = p_1 p_2 \dots p_r \text{ with } p_1, p_2, \dots, p_r \text{ distinct prime numbers} \\ 0, & \text{if } p^2 \mid n \text{ with } p \text{ prime.} \end{cases}$$

Theorem 6: (Möbius inversion formula). If F and f are two number theoretic functions related by the formula

$$F(n) = \sum_{d|n} f(d),$$

then

$$f(n) = \sum_{d|n} \mu(d)F(n/d) = \sum_{d|n} \mu(n/d)F(d).$$

The Partition Function

Definition 6: The partition function [3], denoted by $p(n)$, is the number of ways to write the positive integer n as a sum of positive integers, where the order of the addends is not important.

E.g. $p(5) = 7$, since the partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, and 1 + 1 + 1 + 1 + 1.

4. Multiplicative Function

Definition 7: A number theoretic function f is said to be *multiplicative* if $f(m \cdot n) = f(m) \cdot f(n)$, where m and n are relative primes. A number theoretic function f is said to be *completely multiplicative* if $f(m \cdot n) = f(m) \cdot f(n)$ for all positive integers m and n in the domain of f .

Theorem 7: τ , the number of positive divisors function; σ , the sum of the positive divisors function; ϕ , the Euler Phi function and μ , the Möbius function are multiplicative.

Theorem 8: If f and g are multiplicative functions, then their product fg and quotient f/g ($g \neq 0$) are also multiplicative.

5. Some Relationships among the Number Theoretic Functions

Now the author wants to establish some relationships among the number theoretic functions discussed above.

Theorem 9: The Euler Phi-function ϕ , the number of positive divisors function τ , and the sum of the positive divisors function σ , satisfy the formulae

$$(a) \sum_{d|n} \phi(d) \cdot \tau(n/d) = \sigma(n).$$

$$(b) \sum_{d|n} \phi(d) \cdot \sigma(n/d) = n \cdot \tau(n)$$

$$(c) n \text{ is a prime number if and only if } \sigma(n) + \phi(n) = n \cdot \tau(n).$$

Proof: (a) Let $n = p^a$, where p is a prime and $a > 0$. Then $d = \{1, p, p^2, p^3, \dots, p^a\}$.

$$\sum_{d|n} \phi(d) \cdot \tau(n/d) = \phi(1) \cdot \tau(p^a) + \phi(p) \cdot \tau(p^{a-1}) + \phi(p^2) \cdot \tau(p^{a-2}) + \phi(p^3) \cdot \tau(p^{a-3}) + \dots$$

$$+ \phi(p^{a-1}) \cdot \tau(p) + \phi(p^a) \cdot \tau(1) =$$

$$1 \cdot (a+1) + (p-1) \cdot a + p(p-1) \cdot (a-1) + p^2(p-1) \cdot (a-2) + \dots + p^{a-2}(p-1) \cdot 2 + p^{a-1}(p-1) \cdot 1$$

$$= 1 + p + p^2 + p^3 + \dots + p^a = \sigma(n) \text{ (by Theorem 2 and Theorem 5).}$$

If $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, then, since both τ and ϕ are multiplicative (Theorem 7),

$\phi(n) = \phi(p_1^{a_1})\phi(p_2^{a_2})\dots\phi(p_r^{a_r})$ and $\tau(n) = \tau(p_1^{a_1})\tau(p_2^{a_2})\dots\tau(p_r^{a_r})$. Moreover, the product $\phi(n)\tau(n)$ is also multiplicative (Theorem 8); the result in (a) will hold for such an n .

(b) If $n = p^a$, where p is a prime and $a > 0$, then $d = \{1, p, p^2, p^3, \dots, p^a\}$.

$$\begin{aligned} & \sum_{d|n} \phi(d) \cdot \sigma(n/d) \\ &= \phi(1) \cdot \sigma(p^a) + \phi(p) \cdot \sigma(p^{a-1}) + \phi(p^2) \cdot \sigma(p^{a-2}) + \dots + \phi(p^{a-1}) \cdot \sigma(p) + \phi(p^a) \sigma(1) = \\ &= 1 \cdot (1 + p + p^2 + \dots + p^a) + (p-1)(1 + p + p^2 + \dots + p^{a-1}) + \dots + p^{a-2}(p-1)(1 + p) + p^{a-1}(p-1) \cdot 1 \\ &= p^a + p^a + \dots + p^a = (a+1)p^a = \tau(p^a) \cdot p^a = n \cdot \tau(n) \text{ (by Theorem 1 and Theorem 5).} \end{aligned}$$

If $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, then using the multiplicative property of ϕ and σ , one can show that this result is also true.

(c) If $n = p$ is a prime number, then $\sigma(n) + \phi(n) = \sigma(p) + \phi(p) = p + 1 + p - 1 = 2p$ and $n \cdot \tau(n) = p \cdot \tau(p) = p \cdot (2) = 2p$. Hence, $\sigma(n) + \phi(n) = n \cdot \tau(n)$ when n is a prime.

Conversely, assume $\sigma(n) + \phi(n) = n \cdot \tau(n)$ and n is not a prime. If $n = 12 = 2^2 \cdot 3$, then $\sigma(n) + \phi(n) = \sigma(12) + \phi(12) = 28 + 4 = 32$, but $n \cdot \tau(n) = 12 \cdot 6 = 72$. The two sides are not equal, which is a contradiction. Therefore, n must be a prime for the relation

$$\sigma(n) + \phi(n) = n \cdot \tau(n)$$

to hold.

At this point the author would like to establish a relationship between the Möbius μ function and the Euler Phi-function ϕ . In order to do this, one needs a theorem known as the Gauss Theorem.

Theorem 10: (Gauss) If $n \in \mathbb{Z}$ with $n > 0$, then $\sum_{d|n, d>0} \phi(d) = n$.

Theorem 11: The functions ϕ and μ are related by $\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$.

Proof: By Theorem 10, one has $\sum_{d|n, d>0} \phi(d) = n$ for $n > 0$. If $F(n) = n$,

then applying Theorem 6 on both sides of the above equation one can have

$$\phi(n) = \sum_{d|n} \mu(d) F(n/d) = \sum_{d|n} \mu(d) \cdot \frac{n}{d} = n \sum_{d|n} \frac{\mu(d)}{d}.$$

Now the author wants to show the relationship between τ and μ and between σ and μ .

Theorem 12: (a) The functions τ and μ satisfy the equation $\sum_{d|n} \mu(d) \tau(n/d) = 1$.

(b) The functions σ and μ satisfy the equation $\sum_{d|n} \mu(d) \sigma(n/d) = n$.

Proof:

(a) τ can be written as $\tau(n) = \sum_{d|n} 1$. By applying the Möbius inversion formula, one has

$$\sum_{d|n} \mu(d) \tau(n/d) = 1.$$

(b) σ can be written as $\sigma(n) = \sum_{d|n} d$, which can be written, by using the Möbius inversion formula,

$$\text{as } \sum_{d|n} \mu(d)\sigma(n/d) = n.$$

Theorem 13: The partition function p and the sum of positive divisors function σ are related by the formula [4, 5]

$$np(n) = \sum_{k=0}^n \sigma(k)p(n-k).$$

In Theorem 9, Theorem 11, Theorem 12, and Theorem 13, one can see the relationships between different number theoretic functions. Using these relationships, one can create many other relationships between these functions. For example, using Theorem 9 and Theorem 13, one can have a relation between the partition function p , the Euler function ϕ , and the number of positive divisors function τ . In addition, by using Theorem 12 (part (b)) and Theorem 13, one can have a relation between the Möbius μ function and the partition function p .

6. References

1. G. E. Andrews, *Number Theory*, Dover, 1971.
2. G. H. Hardy, and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Clarendon Press, 1979.
3. G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1998.
4. L. Euler, *Introduction to Analysis of the Infinite*, Vol.1 (Translation by J. D. Blanton), Springer-Verlag, 1988.
5. T. Osler, A. Hassen, and T. R. Chandrupatla, Surprising Connections between Partitions and Divisions, *The College Mathematics Journal*, 38, (2007) 278 -287.