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A Study of the Two-body Problem with Magnetic Interactions

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Abstract

A first order correction is applied to Kepler's third law that takes into account the magnetic interaction between two orbiting bodies. This correction is derived by introducing a magnetic moment term in the potential energy of the system. Deviations of the period, distances, and velocities, from those implied by the classic Kepler's law (based on gravitation only) are derived which may be measurable in astronomical systems such as close binary systems of magnetic stars, and possibly even in extrasolar planetary systems.

Key Words: Kepler's laws, orbits, magnetic moment

1. Introduction

Kepler's third law of planetary motion is well known, in which the period squared of the planet is proportional to the distance cubed. This, of course, is directly derived from Newton's laws, and indeed, can be more generally expressed in terms of any two bodies under the influence of gravity alone.

$$T_e^2 = \left(4\pi^2/GM\right)a^3 \,, \tag{1}$$

where T_e is the period of the elliptical orbit, G is the gravitational constant, M is the sum of the masses, and a is the semi-major axis of the orbit of the reduced mass. One may derive this expression by putting the gravitational potential energy into the virial theorem.

In this paper, we explore the addition of a magnetic energy term (in other words, the two orbiting bodies are also magnetic dipoles) to the total potential energy, and see what first order correction we can derive for the period, size of orbits, and respective velocities.

One might be tempted to dismiss such an exercise as theoretical only, but we would like to caution the reader that there are many known binary star systems whose parameters (orbital radius, mass ratio, etc.) span large ranges; indeed, we know of a number of systems in which the two stars are very close to each other—indeed are gravitationally interacting strongly [1]. Such interacting contact binaries are now thought to be an important part of stellar evolution, at least for a certain class of stars. Gravitational interaction can also be of importance in the evolution of extrasolar planetary systems [2]. Under such circumstances, strong magnetic dipoles can interact to change the orbital characteristics.

Furthermore, we know of many stars (main sequence as well as degenerate) that are strongly magnetized [3,4]. Magnetic moments of stars can vary a great deal, and they can be quite large. Between the considerations of large magnetic moments and close orbits, this calculation can seem quite relevant. But are the results measurable?

If one of the two orbiting bodies is a pulsar, then we have a very accurate time piece that rivals the precision of an atomic clock. Indeed, this was used to great effect to infer the emission of gravitational radiation from the binary pulsar PSR 1913+16 [5,6]. Russell Hulse and Joseph Taylor were awarded the Nobel Prize for the discovery of this pulsar. By measuring the decay of the orbit of this binary system, they were able to show that the system was emitting gravitational radiation consistent with General Relativity.

Furthermore, magnetized stars may give us the opportunity to take accurate radio measurements of the distance of the components from each other, and deviations from that as implied by gravitational attraction alone. Although this would require space based Very Long Baseline Interferometry (VLBI), such technology is nearly at hand.

We currently employ the technology of very high resolution spectroscopy (the relative Doppler shift of absorption lines) to discover extrasolar planets [2]. Other technologies are being developed, but this one has been the dominant tool used for the discovery of hundreds of such systems. The same technology could be used to detect slight changes in velocity consistent with the first order correction to Kepler's law that we derive here in this paper.

Besides deriving the first order (magnetic) correction term to Kepler's third law, we analyze the strength of that term under different circumstances (distance and size of the magnetic moment) to see if indeed this term is detectable. Could it be possible that with a precise enough measurement of period, we can infer the magnetic moments and vice versa for a pair of stars (or planet and star combination)?

2. Basic Equations

Let us begin by exploring the basic equations necessary to investigate a typical binary system with magnetic components. To begin with, we assume a two body system in which each body has a magnetic moment $\mathbf{m}_i$, roughly of equal magnitude $|\mathbf{m}_1| \sim |\mathbf{m}_2| \sim \mathbf{m}$, and more or less anti-aligned with each other, $\mathbf{m}_1 \sim -\mathbf{m}_2$. This is the lowest energy configuration. One might be tempted to use conservation of angular momentum arguments to think that the magnetic moments would be aligned, but just like refrigerator magnets which flip to present an anti-alignment configuration, we have here an ultimately frictionless environment in which the same thing would happen during the formation of a binary system. The anti-alignment wouldn't be perfect, however, and later we will investigate any residual torque that would be experienced by a misaligned system.

a. The magnetic field

The magnetic field of a dipole can be written in coordinate-free form [7] as:

$$\mathbf{B}_{dip}(\mathbf{r}) = \left(\mu_0 / 4\pi \right) \left[3\left(\mathbf{m} \cdot \hat{\mathbf{r}} \right) \hat{\mathbf{r}} - \mathbf{m} \right] / r^3 . \qquad (2)$$

(Here the term $\hat{\mathbf{r}}$ is meant to represent a unit vector in the line joining the two orbiting bodies.) The first term is negligible because we are assuming that each magnetic moment is roughly perpendicular to the line joining the two bodies, so the magnetic field that one body feels due to the presence of the other body is approximately,

$$\mathbf{B}(r) \approx -(\mu_0 / 4\pi) \mathbf{m} / r^3 . \qquad (3)$$

b. Torque

In a misaligned system, let us investigate the kind of torques with which we are dealing. The torque can be represented as,

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} . \qquad (4)$$

Further, if we take from equation (3) that regardless of orientation, the magnetic field is proportional to

$$B \propto m / r^3 , \qquad (5)$$

then we can say to an order of magnitude, for magnetic moments of comparable strength, that

$$N \sim m^2 \sin\theta / r^3 . \qquad (6)$$

Indeed, recognizing that $N = I\, d^2\theta / dt^2$, equation (6) becomes a differential equation that is recognizable as simple harmonic motion. The period of oscillation for this motion is (for small angle),

$$T_{\text{osc}} \approx (2\pi/m)(Ir^3)^{1/2} . \tag{7}$$

So then, the period of oscillation is proportional to,

$$T_{\text{osc}} \propto r^{3/2}/m . \tag{8}$$

c. Scaling

Now we should ask the question as to how the magnetic moment affects scale with size and distance. Clearly, because of conservation of magnetic flux, the magnetic field of a star is inversely proportional to the square of its radius,

$$B \propto 1/R^2 . \tag{9}$$

But we also have from equation (5) that the magnetic field due to a magnetic moment is inversely proportional to the cube of the distance from that magnetic moment. Therefore, combining equations (5) and (9) to investigate how the magnetic moment of a star is related to its radius, we look at the surface magnetic field and find that the magnetic moment is proportional to the radius,

$$m \propto R . \tag{10}$$

Since according to equation (6) the torque is inversely proportional to the distance cubed (and proportional to the magnetic moment squared), and according to equation (10) the magnetic moment is proportional to the radius, it would behoove us to get the pair of orbiting stars as close to each other as possible for maximum torque. Thus, at minimum distance, combining equations (6) and (10), we have that,

$$N \propto 1/R . \tag{11}$$

This being the case, in order to consider the maximum torque possible we need to consider a closely orbiting pair of neutron stars. The largest torque would produce the shortest period. Therefore, to understand the range of values for oscillation period using equation (7), let us consider the extreme case—two neutron stars as a contact binary, and a more typical case—two stars like the sun separated by one astronomical unit. In the former case, the period derived is around 33 seconds, and in the latter case, it is around 100 days. As one can see, for a practical range of possibilities, the periods derived should be in principle observable by astronomers.

d. Magnetic potential energy term

Finally, it is very important that we derive the potential energy term for the interacting magnetic dipole. After all, it is the introduction of this term that will give us the first order correction to Kepler's third law, among other things. Recall that the potential energy of the interaction of a dipole with a magnetic field is,

$$V_B = -(\mathbf{m} \cdot \mathbf{B}) . \tag{12}$$

The magnetic field that the dipole sees has already been described by equation (2). As mentioned before, we assume that angular momentum was conserved through the stellar formation process, so we would expect that the magnetic moment axes would be more or less aligned. That being the case, if we assume equal magnetic moments, the magnetic field reduces to equation (3). So, applying equation (12) to equation (3), and assuming $\cos\theta$ is close to unity, we get,

$$V_B = (\mu_0/4\pi)(\mathbf{m_1} \cdot \mathbf{m_2})/r^3 = (-\mu_0/4\pi)m^2/r^3 \cos\theta \approx (-\mu_0/4\pi)m^2/r^3 . \tag{13}$$

Thus, this is the extra potential energy term that must be included in the total energy and the virial theorem to derive stability, Kepler's third law, etc. So, the total potential energy term, combining gravity and magnetic forces is,

$$V = -G\left(M_1 M_2\right)/r - \left(\mu_0 / 4\pi\right) m^2 / r^3 \quad . \tag{14}$$

3. Conditions for circular orbits and stability of two bodies

The two-body problem under the influence of a gravitational force has been discussed in a number of classical mechanic textbooks [e.g. 8, 9, 10]. In this section we will discuss the condition for circular orbits and the stability of two bodies under both gravitational and magnetic forces. Most treatments of the central force problem assume a force as a single power law term. For instance, it is found that the orbit due to a central force in the form of $f(r) \propto 1/\left(r^{3-\beta^2}\right)$ is a closed, stable orbit if β is a rational number [8]. In this paper the central force has two terms representing both gravitational and magnetic forces. The magnetic force, being a much smaller force compared to the gravitational force, could be considered as a perturbation or higher order term.

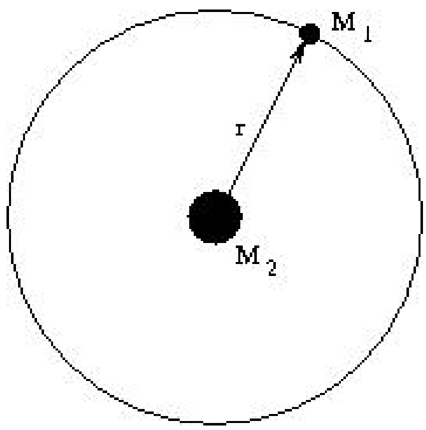

Fig. 1: The binary system in which M_1 is executing a circular orbit of radius r around M_2, $\left(M_1 << M_2\right)$.

Let us first begin by calculating the required speed for a purely gravitational potential in planetary motion. Keeping things simple, for now, suppose mass M_1 is orbiting in a circular path with radius of r_{C0} around M_2 where M_1 << M_2, as shown in Fig 1. Then, by a simple calculation, the required speed for the circular orbit is given by

$$v_0 = \sqrt{\frac{GM_2}{r_{C0}}} \quad . \tag{15}$$

By using the fact that the angular momentum, $l_0 = M_1 r_{C0} v_0$, is a constant of motion, one can determine the radius of circular orbit:

$$r_{C0} = \frac{l_0^2}{GM_1^2 M_2} \quad . \tag{16}$$

According to the Kepler's third law, the relationship between period T_0 and the radius of circular orbit is

$$T_0 = 2\pi\sqrt{\frac{r_{C0}^3}{GM_2}} \, . \tag{17}$$

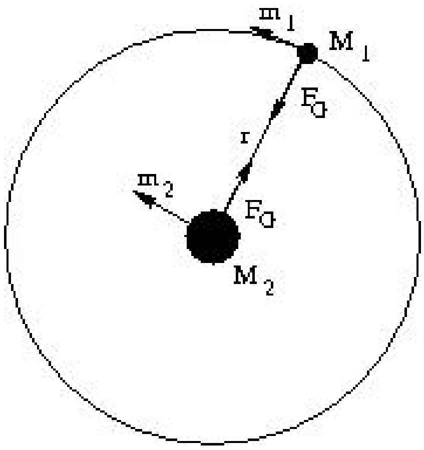

Fig. 2: The binary system showing the respective magnetic moments $\mathbf{m_1}$ and $\mathbf{m_2}$.

Now let us consider the effect of magnetic moments on these two fundamental quantities of the circular orbit (see Fig. 2). The gravitational potential energy is given by

$$V_G = -G\frac{1}{r}M_1M_2 \, , \tag{18}$$

where, as before, M_1 and M_2 are the masses.

The potential energy due to magnetic moments has already been given by equation (13),

$$V_m = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[\left(\mathbf{m_1}\cdot\mathbf{m_2}\right) - 3\left(\mathbf{m_1}\cdot\hat{\mathbf{r}}\right)\left(\mathbf{m_2}\cdot\hat{\mathbf{r}}\right)\right]. \tag{19}$$

For near alignment of magnetic momentum when $\theta_1 \approx \theta_2 \approx \frac{\pi}{2}$, we obtain

$$V_m = -\frac{\mu_0}{4\pi}\frac{1}{r^3}m_1m_2 \, . \tag{20}$$

As previously shown in equation (14), the total potential energy is obtained from combining equations (18) and (20),

$$V = V_G + V_m = \frac{-GM_1M_2}{r} - \frac{\mu_o}{4\pi}\frac{m_1m_2}{r^3} \, . \tag{21}$$

In general, for central forces, the orbit can be an ellipse. The conservation of energy implies the total energy is a constant of the motion:

$$E = (1/2)M_1\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r) \, , \tag{22}$$

where ϕ is the angular position. The second constant of the motion is angular momentum

$$l = M_1 r^2 \dot{\phi}\,. \tag{23}$$

The effective potential is defined by

$$V_{\text{eff}} = V + \frac{l^2}{2M_1 r^2}\,. \tag{24}$$

To reduce this to a one-dimensional problem in r, we can rewrite the total energy in terms of an effective potential,

$$E = (1/2) M_1 \dot{r}^2 + V_{\text{eff}}(r)\,. \tag{25}$$

The condition for a circular stable orbit is $\left.\frac{\partial V_{\text{eff}}}{\partial r}\right|_{r=r_C} = 0$ and $\left.\frac{\partial^2 V_{\text{eff}}}{\partial r^2}\right|_{r=r_C} > 0$, where r_C is the radius of the circular orbit. The first condition implies

$$r_c = \frac{\dfrac{l^2}{2M_1} \pm \sqrt{\left(\dfrac{l^2}{2M_1}\right)^2 + 3(GM_1M_2)\dfrac{\mu_o m_1 m_2}{4\pi}}}{GM_1M_2}\,. \tag{26}$$

Since $3(GM_1M_2)\dfrac{\mu_o m_1 m_2}{4\pi} << \left(\dfrac{l^2}{2M_1}\right)^2$, the minus sign is not practical (it would imply a very small orbital radius, r_C, smaller than the sum of the radii of the two orbiting bodies). Thus

$$r_c = \frac{\dfrac{l^2}{2M_1} + \sqrt{\left(\dfrac{l^2}{2M_1}\right)^2 - 3(GM_1M_2)\dfrac{\mu_o m_1 m_2}{4\pi}}}{GM_1M_2}\,. \tag{27}$$

Note that in the absence of magnetic moments it reduces to the pure gravitational solution, equation (16).

The second condition, that $\left.\frac{\partial^2 V_{\text{eff}}}{\partial r^2}\right|_{r=r_C} > 0$ requires

$$1 - \frac{3\left(\dfrac{\mu_0}{4\pi} m_1 m_2\right)}{\left(\dfrac{l^4}{GM_1^3 M_2}\right)} + \sqrt{1 - \frac{3\left(\dfrac{\mu_0}{4\pi} m_1 m_2\right)}{\left(\dfrac{l^4}{GM_1^3 M_2}\right)}} > 0\,. \tag{28}$$

Since $\dfrac{3\left(\dfrac{\mu_0}{4\pi}m_1m_2\right)}{\left(\dfrac{l^4}{GM_1^3M_2}\right)} < 1$, this condition is obviously satisfied, so we do indeed have stability.

We have the virial theorem as well [8], which states that

$$\overline{KE} = \frac{1}{2}\overline{\frac{\partial V}{\partial r}r}\,. \tag{29}$$

Using the above equation for circular motion, $r = r_C$, we have

$$T = \left[\frac{G}{4\pi^2}\frac{M_2}{r_C^3}\left(1+\frac{3\mu_0}{4\pi G}\frac{m_1m_2}{M_1M_2}\frac{1}{r_C^2}\right)\right]^{-1/2} = T_0\left(1+\frac{3\mu_0}{4\pi G}\frac{m_1m_2}{M_1M_2}\frac{1}{r_C^2}\right)^{-1/2}, \tag{30}$$

where, from equation (17),

$$T_0 = \sqrt{\frac{4\pi^2}{GM_2}r_C^3} \approx \sqrt{\frac{4\pi^2}{GM_2}r_{C0}^3}\,. \tag{31}$$

By linear approximation of equation (30) we get

$$T = T_0 - \left(\frac{3}{2}\frac{\mu_0}{4\pi G}\frac{m_1m_2}{M_1M_2}\frac{1}{r_C^2}\right)T_0\,. \tag{32}$$

Hence,

$$\left|\frac{T-T_0}{T_0}\right| = \frac{3}{2}\frac{\mu_0}{4\pi G}\frac{m_1m_2}{M_1M_2}\frac{1}{r_C^2}\,. \tag{33}$$

For our linear approximation, we use $r_C \approx r_{C0}$, where r_{C0} and r_C are the radii of the circular orbit according to the Kepler's law, and from this modified method, based on respective magnetic moments, we have,

$$\left|\frac{T-T_0}{T_0}\right| = \frac{3}{2}\frac{\mu_0}{4\pi G}\frac{m_1m_2}{M_1M_2}\frac{1}{r_{C0}^2}\text{, or} \tag{34}$$

$$\left|\frac{T-T_0}{T_0}\right| = \frac{3}{2}\frac{G\mu_0}{4\pi}\left(\frac{M_1^3M_2m_1m_2}{l_0^4}\right). \tag{35}$$

Equation (35) represents, then, the degree to which the magnetic moments affect the period of revolution of M_1 around M_2. We can also investigate the fractional change in radius that the presence of magnetic moments would yield. By linear approximation one can find

$$r_{\mathrm{C}} \approx \frac{l^2}{\left(GM_1^2 M_2\right)} - 3\left(\mu_0 m_1 m_2 \middle/ 4\pi\right)\left(M_1 \middle/ l^2\right), \text{ and} \tag{36}$$

$$r_{\mathrm{C}} \approx r_{\mathrm{C0}} - 3\left(\mu_0 m_1 m_2 \middle/ 4\pi\right)\left(M_1 \middle/ l^2\right). \tag{37}$$

Now we can apply the same method for the fractional change of the circular radii,

$$\left|\frac{r_{\mathrm{C}} - r_{\mathrm{C0}}}{r_{\mathrm{C0}}}\right| \approx \left(\frac{3G\mu_0}{4\pi}\right)\frac{\left(M_1^3 M_2 m_1 m_2\right)}{l_0^4}. \tag{38}$$

As we did for period and radius, we can also investigate the fractional change of the velocity, for the introduction of magnetic moments. By using the fact that the velocity $v = \frac{2\pi r_{\mathrm{C}}}{T}$, one obtains by substitution of equations (32) and (33),

$$v \approx \frac{2\pi r_{\mathrm{C0}}\left(1 - \left(\frac{3\mu_0 G}{4\pi}\right)\frac{\left(M_1^3 M_2 m_1 m_2\right)}{l^4}\right)}{T_0\left(1 - \left(\frac{3}{2}\frac{\mu_0}{4\pi G}\frac{m_1 m_2}{M_1 M_2}\frac{1}{r_{\mathrm{C}}^2}\right)\right)}. \tag{39}$$

Once again, solving for the fractional change, this time of velocity, we get

$$\left|\frac{v - v_0}{v_0}\right| \approx \left(\frac{3G\mu_0}{8\pi}\right)\frac{\left(M_1^3 M_2 m_1 m_2\right)}{l_0^4}. \tag{40}$$

Note that the relative differences of period, radius and velocity satisfy the following relation:

$$\frac{r - r_0}{r_0} = \frac{v - v_0}{v_0} + \frac{T - T_0}{T_0}, \tag{41}$$

which can be written as

$$\frac{\delta r}{r_0} = \frac{\delta v}{v_0} + \frac{\delta T}{T_0}. \tag{42}$$

Our derivations [equations (35), (38), and (40)] are based on circular orbits, so there would be little difference in the results for mildly elliptical orbits such as those belonging to our very own solar system. For more extreme elliptical orbits, a more extensive analysis would need to be done, which is beyond the purpose of this study.

Equation (41) can be obtained more directly by recognizing that for circular orbits, the relationship between velocity and period is,

$$v = \frac{2\pi r}{T}, \tag{43}$$

so by differentiation, we get equation (42) directly. What we cannot show directly, however, is that each term is equally weighted, that is, by comparing equation (40) and equation (35), we see that each term in equation (42) contributes the same value,

$$\frac{\delta r}{r_0} = 2\frac{\delta v}{v_0} = 2\frac{\delta T}{T_0}. \quad (44)$$

The technically correct form of Kepler's third law is to use the reduced mass throughout all the calculations. We have derived the aforementioned equations using the extreme case of one mass being much larger than the other, such as a planet orbiting the Sun. The other extreme would be a two body problem of equal masses. The reduced mass of one extreme versus the other differs only by a factor of 2. Since our calculations are only order of magnitude estimates of a first order perturbation calculation, it matters little what the actual reduced mass is, as we shall see later when we introduce characteristic values in these equations. Thus, from equation (35),

$$\left|\frac{T - T_0}{T_0}\right| = \frac{3}{2}\frac{\mu_0}{4\pi G}\frac{m_1 m_2}{M_1 M_2}\frac{1}{R^2_{\text{Sun}}}\frac{1}{\left(r_{\text{C}}/R_{\text{Sun}}\right)^2}, \text{ or} \quad (45)$$

$$\left|\frac{T - T_0}{T_0}\right| = \left|\frac{\delta T}{T_0}\right| = \alpha\frac{1}{\left(r_{\text{C}}/R_{\text{Sun}}\right)^2}, \quad (46)$$

where $\alpha = \frac{3}{2}\frac{\mu_0}{4\pi G}\frac{m_1 m_2}{M_1 M_2}\frac{1}{R^2_{\text{Sun}}}$.

Using $\mu_0 = 4\pi\times10^{-7}\ \frac{\text{T}^2\text{m}^3}{\text{J}}$, $G = 6.673\times10^{-11}\ \frac{\text{m}^3}{\text{kg.s}^2}$ and $R_{\text{Sun}} = 6.95\times10^8\ \text{m}$,
we obtain

$$\alpha = 4.65\times10^{-15}\left(\frac{\text{T}^2\ \text{kg s}^2}{\text{J m}^2}\right)\frac{m_1 m_2}{M_1 M_2}. \quad (47)$$

In the case of our Sun, a pretty typical main sequence star, the magnetic moment is 3.36×10^{29} J/T, which means that for two Sun-like stars orbiting each other, $\alpha = 1.31\times10^{-16}$. Naturally, this value would be orders of magnitude smaller for degenerate objects such as white dwarfs and neutron stars, and orders of magnitude larger for red giants and supergiants.

From equations (46) and (47), we see the effect on the period by the presence of the magnetic moments, as shown in Fig. 3.

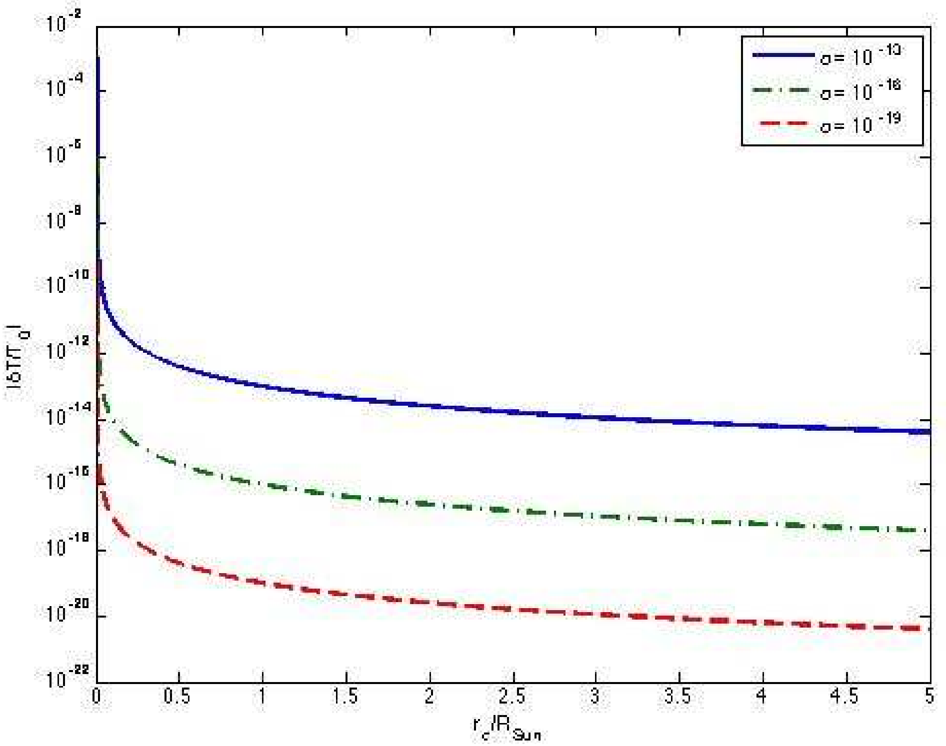

Fig. 3: Fractional change in period versus radius for $\alpha = 10^{-13}, 10^{-16}$ and 10^{-19} [see equations (46) and (47)].

4. Conclusion

It would be interesting to see if this is merely a purely theoretical calculation, or whether the derived fractional changes shown in equations (32), (36), and (38) are indeed measurable. Let us look at the fractional change in period, as described by equation (32), because this should be the easiest parameter to measure. Clearly it would be helpful for our stars to have large magnetic moments, which would imply large magnetic fields. It has been shown that main sequence stars can have magnetic fields as high as three orders of magnitude larger than the Sun [3]. Given this, we could imagine a not very extreme scenario of two red dwarfs, each of 0.1 solar masses and of high magnetic field (0.2 T), orbiting each other at a distance of 0.1 AU. This would yield a change in period over the course of a single year of 166 seconds. This effect, then, is clearly measurable.

Therefore, we must conclude that although the circumstance may be a bit rare, there must exist binary systems whose deviation from a strictly gravitational expression of Kepler's Law are measurable with current technology. We therefore recommend that those with the opportunity look for such systems to measure.

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