

**Certain Rings and Semigroups Examining the Regularity Property**

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**Abstract**

A number of main properties of the commuting regular rings and commuting regular semigroups have been studied in this paper. Some significant results of which will be used for the commutative rings and a necessary and sufficient condition is given for a semigroup to be commuting regular.

**1. Introduction**

We use  $R$  and  $S$  to denote a ring and a semigroup, respectively. The nilradical of the ring  $R$  denoted by  $N(R)$ . An element  $a$  of a semigroup  $S$  is called regular if there exists  $x$  in  $S$  such that  $axa = a$ . The semigroup  $S$  is called regular if all its elements are regular. For elements  $a$  of a semigroup  $S$ , we say that  $b$  is an inverse of  $a$  if both of the relations  $aba = a$  and  $bab = b$  hold. The set of inverses of an element  $a \in S$ , denote by  $V(a)$ . (See [3] ) A ring  $R$  is called commuting regular ([6] and [7]) if and only if for each  $x, y \in R$  there exists an element  $z$  of  $R$  such that  $xy = yxzyx$ . The commuting regular semigroup have been defined in a similar way in [1]. Following [5], a nonempty subset  $T$  of a ring  $R$  is multiplicative provided that,  $a, b \in T$  implies  $ab \in T$ . Let  $T$  be a multiplicative subset of a commutative ring  $R$ , the ring of quotients of  $R$  by  $T$  denoted by  $T^{-1}R$ . We define a nonempty subset  $T$  of a ring  $R$  (or semigroup  $S$ ) to be commuting regular, if for each  $x, y \in T$  there exists an element  $t$  of  $T$  such that  $xy = yxtyx$ .

**2. Some properties for commuting regular rings**

**Proposition 2.1.** *Let  $R$  be a commutative and commuting regular ring and  $T$  be a multiplicative subset of  $R$ . Then, if  $T$  be a commuting regular subset of  $R$ , then  $T^{-1}R$  is a commuting regular ring.*

**Proof.** Let  $\alpha = \frac{r_1}{s_1}$  and  $\beta = \frac{r_2}{s_2}$  be arbitrary elements in  $T^{-1}R$ , where  $r_1, r_2 \in R$  and  $s_1, s_2 \in T$ , then there

are  $t_1 \in R$  and  $t_2 \in T$  such that  $r_1 r_2 = r_2 r_1 t_1 r_2 r_1$  and  $s_1 s_2 = s_2 s_1 t_2 s_2 s_1$ , so,

$$\alpha\beta = \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2} = \frac{r_2 r_1 t_1 r_2 r_1}{s_2 s_1 t_2 s_2 s_1} = \frac{r_2}{s_2} \cdot \frac{r_1}{s_1} \cdot \frac{t_1}{t_2} \cdot \frac{r_2}{s_2} \cdot \frac{r_1}{s_1} = \beta\alpha\gamma\beta\alpha,$$

where  $\gamma = \frac{t_1}{t_2} \in T^{-1}R$ .

**Proposition 2.2.** *Let  $R$  be a commuting regular ring. If  $I$  be an ideal of  $R$ , then  $R/I$  is a commuting regular ring.*

**Proof.** Suppose that  $a + I, b + I \in R/I$  where  $a, b \in R$ . By the hypothesis there exists  $x \in R$  such that  $ab = baxba$ . Hence  $(a + I)(b + I) = (b + I)(a + I)(x + I)(b + I)(a + I)$ .

**Proposition 2.3.** *Let  $R$  be a commutative ring with identity. If  $R$  is a commuting regular, then every prime ideal of  $R$  is a maximal ideal.*

**Proof.** Let  $P$  be a prime ideal of  $R$ , then  $R/P$  is a commuting regular ring by the Proposition 2.2. If  $0 \neq a \in R/P$ , there exists  $b \in R/P$  such that  $a^2 = a^2ba^2$  and so  $a^2(1-ba^2) = 0$ . Therefore  $1-ba^2 = 0$  or  $ba^2 = 1$  and  $ba = a^{-1}$ . So  $R/P$  is a field and  $P$  is a maximal ideal of  $R$ .

As a useful result of this Proposition we can see  $\dim(R)=0$ .

**Proposition 2.4.** Suppose that  $\{R_\alpha\}_{\alpha \in I}$  is an arbitrary family rings with identity where, each  $R_\alpha$  is commutative and commuting regular. Then  $N(\prod_{\alpha \in I} R_\alpha) = \prod_{\alpha \in I} N(R_\alpha)$ .

**Proof.** By the Proposition 3.1 of [1],  $R = \prod_{\alpha \in I} R_\alpha$  is commuting regular and by the Proposition 2.3,  $\dim R = 0$ . So, by the Proposition 3.1 of [2], we get  $N(R) = \prod_{\alpha \in I} N(R_\alpha)$ .

Recall the following definition from [2]:

**Definition 2.5.** Let  $R$  be a ring and  $x \in N(R)$ . The integer  $\eta(x) = k$  is define to be the index of nilpotency of  $x$ , if  $x^k = 0$  but  $x^{k-1} \neq 0$ . Also,  $\eta(R)$  is defined to be  $\sup\{\eta(x) : x \in N(R)\}$ , if the set  $\{\eta(x) : x \in N(R)\}$  is bounded, and  $\eta(R) = \infty$ , otherwise.

**Theorem 2.6.** Suppose that  $\{R_\alpha\}_{\alpha \in I}$  is an arbitrary family of the commutative and commuting regular rings with identity. Then there exists a positive integer  $k$  such that  $\{\alpha \in I : \eta(R_\alpha) > k\}$  is finite.

**Proof.** Let the assertion fails, then there is an infinite subset  $\{\alpha_i\}_1^\infty$  of  $I$  and elements  $x_{\alpha_i} \in N(R_{\alpha_i})$  such that  $\eta(x_{\alpha_i}) > i$ , so  $x_{\alpha_i} \neq 0$ . Let  $x = \{x_\alpha\} \in \prod_{\alpha \in I} N(R_\alpha)$ , where  $x_\alpha = 0$  if  $\alpha \in I \setminus \{\alpha_i\}_1^\infty$ . Clearly  $x^i = \{x_\alpha^i\} \neq 0$  for each  $i$  and hence  $x \notin N(\prod_{\alpha \in I} (R_\alpha))$  and  $N(\prod_{\alpha \in I} (R_\alpha)) \neq \prod_{\alpha \in I} N(R_\alpha)$ . This contradicts 2.4.

**Proposition 2.7.** Let  $R$  be a commuting regular ring with identity and  $P$  be an ideal in  $R$ . Then if  $P$  is prime then for all  $a, b \in R$ ,  $ab \in P$  yields  $a \in P$  or  $b \in P$ .

**Proof.** Let  $P$  be a prime ideal and  $a, b \in R$  such that  $ab \in P$ . For an arbitrary element  $x \in R$ , there exists an element  $t \in R$  such that  $abx = (bx)at(bx)a \in bRa$ . So,  $abR \subseteq bRa$ . Also, for every  $bx \in bRa$ , there exists  $c \in R$  such that  $bx = a(bx)ca(bx) = ab(xcabx) \in abR$  and then  $bRa \subseteq abR$ . Consequently,  $bRa = abR \subseteq P$  which yields  $a \in P$  or  $b \in P$ .

**Remark.** Commuting regularity is necessary for the above Proposition.

**Proposition 2.8.** Let  $e$  be an idempotent element of a ring  $R$ . If  $R$  is a commuting regular ring, then  $R' = eRe$  is a commuting regular ring with identity.

**Proof.** Clearly  $R'$  is a ring with identity. Let  $exe$  and  $eye$  are arbitrary elements in  $R'$ . Then there are  $t_1, t_2, t_3$  and  $t_4$  in  $R$  such that

$$\begin{aligned}
(exe)(eye) &= e(xey)e = e(yxet_1yxe)e \\
&= e(y(ext_2ex)t_1y(ext_2ex))e \\
&= eyex(t_2ext_1yext_2)exe \\
&= eyex((et_2t_3et_2)xt_1(ext_2yt_3ext_2y))exe \\
&= eyexe(t_2t_3et_2xt_1ext_2yt_3)(xt_2et_4xt_2e)yexe \\
&= (eye)(exe)(et_5e)(eye)(exe),
\end{aligned}$$

where,  $t_5 = t_2t_3et_2xt_1ext_2yt_3xt_2et_4xt_2$ .

### 3. Some properties for commuting regular semigroups

Recall the following definition from [3]:

**Definition 3.1.** A semigroup  $S$  is called a rectangular band if  $aba = a$  for all  $a, b$  in  $S$ .

**Proposition 3.2.** Let  $S$  be a rectangular band semigroup. Then  $S$  is commutative if and only if  $S$  is commuting regular.

**Proof.** Let  $S$  be commuting regular semigroup and  $a, b \in S$ . So there exists an element  $c \in S$  such that  $ab = bacba = ba$ , for,  $S$  is rectangular band. Conversely, let  $S$  be a commutative semigroup, so  $ab = ba = bacba$ , for every  $c \in S$ .

Recall the following definition from [3]:

**Definition 3.3.** Let  $S$  be a semigroup. A relation  $R$  on the set  $S$  is called compatible if

$$(\forall s, t, a \in S) [(s, t) \in R \text{ and } (s', t') \in R] \Rightarrow (ss', tt') \in R.$$

A compatible equivalence relation is called congruence.

Let  $\rho$  be a congruence on a semigroup  $S$  and  $\frac{S}{\rho}$  be the set of  $\rho$ -classes, whose elements are the subsets

$x\rho$ , then we can define a binary operation on the quotient set  $\frac{S}{\rho}$ , in a natural way as follows:

$$(a\rho)(b\rho) = (ab)\rho.$$

It is easy to check that  $\frac{S}{\rho}$  and the above operation is a semigroup.

**Proposition 3.4.** Let  $\rho$  be a congruence on commuting regular semigroup  $S$ . Then  $\frac{S}{\rho}$  is a commuting regular semigroup.

**Proof.** Let  $x, y \in S$ , so there is  $c \in S$  such that  $xy = yxcyx$  and therefore

$$(x\rho)(y\rho) = (xy)\rho = (yxcyx)\rho = (y\rho)(x\rho)(c\rho)(y\rho)(x\rho).$$

Thus  $S$  is commuting regular semigroup.

**Definition 3.5.** Let  $S$  be a semigroup. The left map  $\lambda : S \rightarrow S$  is called a left translation of  $S$  if  $\lambda(st) = (\lambda s)t$ , for all  $s, t$  in  $S$ . The right map  $\rho : S \rightarrow S$  is called a right translation of  $S$  if  $(st)\rho = s(t\rho)$ , for all  $s, t$  in  $S$ . A left translation  $\lambda$  and a right translation  $\rho$  are said to be linked if  $s(\lambda t) = (s\rho)t$  for all  $s, t$  in  $S$ . The set of all linked pairs  $(\lambda, \rho)$  of left and right translation is called the translation hull of  $S$  and will be denoted by  $\Omega(S)$ .  $\Omega(S)$  is a semigroup under the obvious multiplication  $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$  where  $\lambda\lambda'$  denote the composition of the left maps  $\lambda$  and  $\lambda'$ , while  $\rho\rho'$  denotes the composition of the right maps  $\rho$  and  $\rho'$ .

**Proposition 3.6.** Let  $S$  be a commuting regular semigroup. For every  $a$  in  $S$  define  $\lambda_a s = as$  and  $s \rho_a = sa$  ( $s \in S$ ). Then  $(\lambda_a, \rho_a)$  is a linked pair in  $\Omega(S)$  and the set of every  $(\lambda_a, \rho_a)$ , where  $a \in S$ , with multiplication of link translations is a commuting regular semigroup.

**Proof.** It is easy to verify that, for all  $a, b$  in  $S$ ,  $(\lambda_a, \rho_a)(\lambda_b, \rho_b) = (\lambda_{ab}, \rho_{ab})$  and the set of every  $(\lambda_a, \rho_a)$ , with above multiplication, is a semigroup and if  $a, b \in S$ , then there is  $t \in S$  such that  $ab = batba$  so

$$(\lambda_a, \rho_a)(\lambda_b, \rho_b) = (\lambda_{ab}, \rho_{ab}) = (\lambda_{batba}, \rho_{batba}) = (\lambda_b, \rho_b)(\lambda_a, \rho_a)(\lambda_t, \rho_t)(\lambda_b, \rho_b)(\lambda_a, \rho_a).$$

**Proposition 3.7.** Let  $S$  be a commuting regular semigroup with the set  $E$  of the idempotents. Then  $E$  is a regular subsemigroup of  $S$ . Moreover, for every element  $a$  of  $E$ ,  $a \in V(a)$ .

**Proof.** Suppose that  $a \in S$ . There exists an element  $s \in S$  such that  $a^2 = a^2 s a^2$ . If  $b = a^2 s$ , then we get

$$b^2 = (a^2 s)(a^2 s) = (a^2 s a^2) s = a^2 s = b,$$

which shows that  $E$  is non empty. For elements  $x$  and  $y$  in  $E$  there exists an element  $t$  of  $S$  such that  $xy = yxtyx$ . Hence,

$$\begin{aligned} (xy)^2 &= (xy)(yxtyx) = x(yy)xtyx \\ &= x(yxtyx) = x(xy) = (xx)y = xy. \end{aligned}$$

Consequently  $E$  is closed under the multiplication of the semigroup and  $x \cdot x \cdot x = x$  yields  $E$  is a regular subsemigroup of  $S$ , moreover,  $x \in V(x)$ .

**Corollary 3.8.** Let  $S$  be a commuting regular semigroup and  $a, b \in S$ . Then  $a'ea$  and  $aea'$  are idempotents, for every  $a' \in V(a)$  and  $e \in E$ , where,  $E$  is the set of idempotents of  $S$ .

**Proof.** For every  $a' \in V(a)$ ,  $(aa')(aa') = (aa'a)a' = aa'$ . Considering the Proposition 3.7 we get  $ea'a' \in E$  and,

$$(a'ea)(a'ea) = (a'ea)(a'e)(aa'a) = a'(ea'a')^2 a = a'ea'a' = a'ea.$$

We use a similar proof to show that  $aea'$  is idempotent.

**Theorem 3.9.** Let  $S$  be a commuting regular semigroup with the set  $E$  of the idempotents. Let  $e, f \in E$  and  $a, b \in S$ . We define the set  $S(e, f)$ , by

$$S(e, f) = \{ g \in V(ef) \cap E : ge = fg = g \}.$$

Then,  $S(e, f)$  is a regular subsemigroup of  $S$ .

**Proof.** By the commuting regularity of  $S$ , there exists an element  $t \in S$  such that  $ef = (fe)t(fe)$ . Then,

$$\begin{aligned} (ef)^2 &= (fetfe)ef = fetf(ee)f = (fetfe)f = (ef)f = e(ff) = ef, \\ ef(ef)ef &= ef(ef)^2 = ef ef = ef, \\ (ef)e &= (fetfe)e = fetf(ee) = fetfe = ef, \\ f(ef) &= f(fetfe) = (ff)(etfe) = fetfe = ef. \end{aligned}$$

This yields,  $ef \in S(e, f)$  and  $S(e, f) \neq \emptyset$ . Let  $x, y \in S(e, f)$ . So there exists an element  $u \in S$  such that  $xy = yxuyx$  (for,  $S$  is commuting regular.) Then,

$$\begin{aligned} (xy)^2 &= xy(yxuyx) = x(yy)xuyx = x(yxuyx) = x(xy) = xxy = xy, \\ ef(xy)ef &= ef(xe)(fy)ef = (efxef)(yef) = efyef = ef, \\ xy(ef)xy &= x(ye)(fx)y = xyxy = xy. \end{aligned}$$

Hence,  $xy \in V(ef) \cap E$ . On the other hand,  $(xy)e = x(ye) = xy$ ,  $f(xy) = (fx)y = xy$ . Let  $x \in S(e, f)$ , then  $x(ef)x = (xe)(fx) = x^2 = x$ . This shows that,  $S(e, f)$  is a regular subsemigroup of  $S$ .

**Proposition 3.10.** *Let  $S$  be a commuting regular semigroup and  $a, b \in S$ . Then,  $b'g a' \in V(ab)$ , for every,  $b' \in V(b)$ ,  $a' \in V(a)$  and  $g \in S(a'a, bb')$ .*

**Proof.** Let  $b' \in V(b)$ ,  $a' \in V(a)$  and  $g \in S(a'a, bb')$ . So,

$$\begin{aligned} ab(b'ga')ab &= (aa'a)(bb')ga'a(bb'b) \\ &= a(a'abb'ga'abb')b \\ &= a(a'abb')b \quad (\text{for, } g \in S(a'a, bb')) \\ &= (aa'a)(bbb) = ab. \end{aligned}$$

And also,

$$(b'ga')(ab)(b'ga') = b'(g(a'abb')g)a' = b'ga' \quad (\text{for, } g \in S(e, f)).$$

Consequently  $b'ga' \in V(ab)$ .

**Corollary 3.11.** *Let  $S$  be a commuting regular semigroup and  $a, b \in S$ . Then  $V(b)V(a) \subseteq V(ab)$ .*

**Proof.** Let  $b' \in V(b)$  and  $a' \in V(a)$ . So,  $(a'a)^2 = a'a$ ,  $(bb')^2 = bb'$  and

$$b'a' = (b'bb')(a'aa') = b'(bb'a'a)a'.$$

Then, by the Theorem 3.9 we get  $bb'a'a \in S(a'a, bb')$ . We now get  $b'a' \in V(ab)$  by using the Proposition 3.10.

**Proposition 3.12.** *Let  $S$  be a commuting regular semigroup with the set  $E$  of the idempotents. Then  $V(e) \subseteq E$ , for every  $e$  in  $E$ .*

**Proof.** Let  $x \in V(e)$ . Then  $xex = e$  and  $exe = x$ . Since  $xe$  and  $ex$  are both idempotent elements then, each one is its inverse (as semigroup element). By the corollary 3.11,  $(ex)(xe)$  is an inverse of  $(xe)(ex)$ , that is to say  $ex^2e$  is an inverse of  $xe^2x = xex = x$ . Consequently,  $x = x(ex^2e)x = (xex)^2 = x^2$ . This shows that  $x \in E$ .

#### 4. Green's Equivalences and commuting regularity

Firstly, we recall the following definitions from [2]:

**Definition 4.1.** If  $a$  is an element of a semigroup  $S$ , the smallest left ideal of  $S$  containing  $a$  is  $Sa \cup \{a\}$  and denoted by  $S^l a$ . An equivalence  $L$  on  $S$  is define by the rule that  $a L b$  if and only if  $S^l a = S^l b$ . Similarly, we define the equivalence  $\mathfrak{R}$  by the rule that  $a \mathfrak{R} b$  if and only if  $aS^l = bS^l$ .

**Proposition 4.2.** Let  $a$  and  $b$  are elements of a semigroup  $S$ . Then  $a L b$  if and only if there exist  $x$  and  $y$  in  $S^l$  such that  $xa = ba$  and  $yb = a$ . Also,  $a \mathfrak{R} b$  if and only if there exist  $u$  and  $v$  in  $S^l$  such that  $au = b$ ,  $bv = a$ .

**Proof.** See [3], Proposition 2.1.1.

**Proposition 4.3.** *The relations  $L$  and  $\mathfrak{R}$  commute.*

**Proof.** See [3], Proposition 2.1.2.

**Proposition 4.4.** *Let  $S$  be a commuting regular semigroup and  $a, b \in S$ . Then  $a L b$  if and only if  $a \mathfrak{R} b$ .*

**Proof.** Suppose that  $a L b$ . By the Proposition 4.2, there are  $x$  and  $y$  in  $S$  such that  $xa = ba$  and  $yb = a$ . So, there are  $t_1, t_2$  in  $S$  such that

$$a = xa = ax t_2 a x = u x, \quad b = y b = b y t_2 b y = v y,$$

where,  $u = a x t_1 a$  and  $v = b y t_2 b$ . Now, using the last part of the Proposition 4.2 gives us  $a \mathfrak{R} b$ . Proof of the converse is Similar.

We define the equivalence  $D$  by

$$D = L \circ \mathfrak{R} = \mathfrak{R} \circ L.$$

The equivalence  $D$  is a two-sided analogue of  $L$  and  $\mathfrak{R}$ . Also we define the equivalence  $\mathfrak{S}$  by the rule  $a \mathfrak{S} b$  if and only if  $S^1 a S^1 = S^1 b S^1$ , that is, if and only if there exist  $x, y, u$  and  $v$  in  $S^1$  such that  $x a y = b$ ,  $u b v = a$ .

It is immediate that  $L \subseteq \mathfrak{S}$  and  $\mathfrak{R} \subseteq \mathfrak{S}$ . Hence, since  $D$  is the smallest equivalence containing  $L$  and  $\mathfrak{R}$ , we get  $D \subseteq \mathfrak{S}$ .

**Proposition 4.5.** *If  $S$  is a commuting regular semigroup, then  $D = \mathfrak{S}$ .*

**Proof.** It is enough to show  $\mathfrak{S} \subseteq D$ . For elements  $a$  and  $b$  in  $S$  let  $a \mathfrak{S} b$ . Then, there are  $x, y, u$  and  $v$  in  $S^1$  such that

$$x a y = b, \quad u b v = a.$$

So, there exists an element  $t_1$  in  $S$  such that

$$a = u b v = u (x a y) v = (y v) (u x a) t_1 (y v) (u x a) = w_1 c,$$

where,  $w_1 = y v u x a t_1 y v u$  and  $c = x a$ , i.e.,  $a L c$  (by the Proposition 4.2). Combining the relations  $x a y = b$  and  $c = x a$  we get  $c y = b$ . Then there exists an element  $t_2 \in S$  such that

$$c = x a = x (u b v) = (x u) (b v) = (b v) (x u) t_2 (b v) (x u) = b w_2,$$

where,  $w_2 = v x u t_2 b v x u$ . This shows that  $c \mathfrak{R} b$ . Then  $\mathfrak{S} \subseteq D$ .

## 5. Conclusions.

Commuting regularity is defined basically for non-commutative rings. Generalizing this property for semigroups and getting necessary and sufficient condition for a semigroup to be commuting regular was the main purpose of this paper.

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