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Composition of two Lorentz boosts through spatial and space-time rotations

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Abstract

We develop the Lorentz matrix in two-dimensional space by two methods consisting of three parameters each. By comparing the two and utilizing the natural invariant properties of the elements, we evolve the composition of two Lorentz boosts. The development specifies how to switch from one three-parameter set to the other. A numerical tool to combine two planar boosts is provided. We extend the method to three spatial dimensions by confining the boosts to a plane. Thus we show, without assuming the existence of a four-dimensional continuum, that the ordered composition of two Lorentz boosts in three-dimensional space yields a Lorentz boost.

Key Words: Lorentz transformation, special relativity, composition of boosts

1. Introduction

The superimposition of two Lorentz boosts yielding a resultant Lorentz boost is normally shown to be a consequence of the existence of such boosts as vectors in a four-dimensional vector space. Such a four-dimensional vector space is also conceived to be the space-time continuum [1, p. 150]. In this paper we show that the superimposition of two Lorentz boosts yields a single resultant Lorentz boost without assuming the presence of a space-time continuum.

The velocity addition formula for co-linear velocities (that is, velocities in the same direction) is easily established in the theory of Special Relativity by considering the transformation of the coordinates of a moving object P as observed from two inertial frames K and K' which themselves are in relative motion [1 pp. 68-69, 2]. This leads to the well-known formula of co-linear velocity addition $(u+v)/(1 + uv/c^2)$. Møller extended this approach to develop the velocity addition formula for combining two velocities in different directions in planar motion [3]. It is also well known that the composition of two boosts in planar motion involves an additional rotation of the line of motion. While the co-linear composition of velocities is commutative, the non-colinear composition of velocities is a non-commutative operation and the order in which the velocities are combined affects the resultant velocity [4].

We describe a method for combining two boosts which are arbitrarily oriented in 3-dimensional space. It should be noted that the orientation of the second boost is well-defined only from the point of view of an observer co-moving with the first boost. This makes it necessary to perform the second boost from an inertial frame K', co-moving with the first boost and in the context of the synchronicity that is unique to the inertial frame K'. These considerations make the composition operation of two boosts non-commutative [4, 5].

We first develop a method for the ordered composition of two arbitrarily oriented planar boosts. We extend this to 3-dimensions by performing a spatial rotation at the outset that confines the two boosts to a plane. The contribution of this paper is the development of a constructive method for the composition of planar boosts, along with a numerical tool and an extension of the method for the composition of three-dimensional boosts.

2. Composition of planar boosts

2.1 The LRL transformation: Two boosts in different directions in a plane

In this section we derive the transformation matrix for a conventional Lorentz ($\mathbf{L}_{xu}$) followed by a planar rotation of the *x*-*y* plane ($\mathbf{R}_{\theta}$) and then followed by another conventional Lorentz ($\mathbf{L}_{xv}$). For clarity, we may visualize four inertial reference frames K, L, M, and N in two-dimensional space. Frames K and L have both their coordinate axes aligned and L is moving at a velocity *u* along the *x*-axis as observed by K. The inertial frame L has another coordinate reference frame M, where the axes of M are rotated by an angle θ counterclockwise with respect to L. Frames M and N have both their coordinate axes aligned and N is moving at a velocity *v* along the *x*-axis as observed by M. The event coordinate transformation from K to N is given by matrix G which is equal to the matrix product $\mathbf{L}_{xv}\,\mathbf{R}_{\theta}\,\mathbf{L}_{xu}$. The matrices $\mathbf{R}_{\theta}$ and $\mathbf{L}_{xv}$ are as specified in equations (1) and (2) respectively, using the notation described in [6].

$$\begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} \qquad (1)$$

$\mathbf{R}_{\theta}$ denotes the above transformation, indicating spatial rotation anticlockwise by an angle θ.

$$\begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & -v\gamma \\ 0 & 1 & 0 \\ -v\gamma/c^2 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} \qquad (2)$$

$\mathbf{L}_{xv}$ denotes a Lorentz transformation of magnitude *v* along the *x*-axis.

The matrix $\mathbf{L}_{xu}$ is the same as the one in equation (2) with velocity *u* substituted for *v*. The inverse of this operation is given by H $= \mathbf{L}_{x(-u)}\,\mathbf{R}_{(-\theta)}\,\mathbf{L}_{x(-v)}$. Matrices G and H are obtained reciprocally from each other by replacing *u* by –*v*, *v* by –*u* and *θ* by –*θ*. Matrices G and H are inverse of each other as the processes associated with them are inverse of each other. The elements of matrices G and H are obtained by matrix multiplication of the corresponding matrices specified in equations (1) and (2) with appropriate values for *v* and θ. Matrices G and H turn out as follows.

$$G = \mathbf{L}_{xv}\,\mathbf{R}_{\theta}\,\mathbf{L}_{xu} = \begin{pmatrix} \gamma_v & 0 & -v\gamma_v \\ 0 & 1 & 0 \\ -v\gamma_v/c^2 & 0 & \gamma_v \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u & 0 & -u\gamma_u \\ 0 & 1 & 0 \\ -u\gamma_u/c^2 & 0 & \gamma_u \end{pmatrix}$$

$$= \begin{pmatrix} \left(\cos\theta + \dfrac{uv}{c^2}\right)\gamma_u\gamma_v & \gamma_v\sin\theta & -\gamma_v\gamma_u\left(v + u\cos\theta\right) \\ -\gamma_u\sin\theta & \cos\theta & u\gamma_u\sin\theta \\ \dfrac{-\gamma_u\gamma_v}{c^2}\left(u + v\cos\theta\right) & \dfrac{-v\gamma_v\sin\theta}{c^2} & \gamma_u\gamma_v\left(1 + \dfrac{uv\cos\theta}{c^2}\right) \end{pmatrix} \qquad (3)$$

$$H = \mathbf{L}_{x(-u)}\,\mathbf{R}_{(-\theta)}\,\mathbf{L}_{x(-v)} = \begin{pmatrix} \left(\cos\theta + \frac{uv}{c^2}\right)\gamma_u\gamma_v & -\gamma_u \sin\theta & \gamma_v\gamma_u(u + v\cos\theta) \\ \gamma_v \sin\theta & \cos\theta & v\gamma_v \sin\theta \\ \frac{\gamma_u\gamma_v}{c^2}(v + u\cos\theta) & \frac{-u\gamma_u \sin\theta}{c^2} & \gamma_u\gamma_v\left(1 + \frac{uv\cos\theta}{c^2}\right) \end{pmatrix} \quad (4)$$

where $\gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}$ and $\gamma_v = \frac{1}{\sqrt{1-v^2/c^2}}$.

The matrix G can be visualized as the composition of *u* and *v* in that order with an orientation shift of θ in between. Similarly, the matrix H can be visualized as the composition of $-v$ and $-u$ in that order with an orientation shift of $-\theta$ in between.

2.2 The RLR transformation: A planar boost preceded and followed by planar rotations

In this section we derive the transformation matrix for RLR, that is, a spatial rotation ϕ followed by a conventional Lorentz and then followed by another spatial rotation α as described in [6].

$$\mathbf{R}_{\alpha}\,\mathbf{L}_{xw}\,\mathbf{R}_{\Phi} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_w & 0 & -w\gamma_w \\ 0 & 1 & 0 \\ -w\gamma_w/c^2 & 0 & \gamma_w \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We denote this transformation matrix as D whose elements are as follows:

$$D = \begin{pmatrix} \gamma_w \cos\alpha\cos\phi - \sin\alpha\sin\phi & \gamma_w \cos\alpha\sin\phi + \sin\alpha\cos\phi & -w\gamma_w \cos\alpha \\ -\gamma_w \sin\alpha\cos\phi - \cos\alpha\sin\phi & -\gamma_w \sin\alpha\sin\phi + \cos\alpha\cos\phi & w\gamma_w \sin\alpha \\ \frac{-w\gamma_w \cos\phi}{c^2} & \frac{-w\gamma_w \sin\phi}{c^2} & \gamma_w \end{pmatrix} \quad (5)$$

where $\gamma_w = \frac{1}{\sqrt{1-w^2/c^2}}$.

Similar to the case with matrices G and H, the inverse of matrix D, corresponding to the transformation $\mathbf{R}_{(-\Phi)}\,\mathbf{L}_{x(-w)}\,\mathbf{R}_{(-\alpha)}$ is denoted as matrix E with the following elements:

$$E = \begin{pmatrix} \gamma_w \cos\phi\cos\alpha - \sin\phi\sin\alpha & -\gamma_w \cos\phi\sin\alpha - \sin\phi\cos\alpha & w\gamma_w \cos\phi \\ \gamma_w \sin\phi\cos\alpha + \cos\phi\sin\alpha & -\gamma_w \sin\phi\sin\alpha + \cos\phi\cos\alpha & w\gamma_w \sin\phi \\ \frac{w\gamma_w \cos\alpha}{c^2} & \frac{-w\gamma_w \sin\alpha}{c^2} & \gamma_w \end{pmatrix} \quad (6)$$

3. Comparison of the LRL and RLR Planar Transformations

3.1 Some characteristics of the LRL and RLR transformation matrices

The LRL matrices G and H described in equations (3) and (4) have a resultant velocity, *w*, whose magnitude can be evaluated by considering the motion of the origin of N *(x′ = 0; y′ = 0)* as

$$w = \frac{\sqrt{u^2 + v^2 + 2uv\cos\theta - u^2v^2\,\frac{\sin^2\theta}{c^2}}}{\frac{uv\cos\theta}{c^2} + 1} . \qquad (7)$$

Equation (7) has also been derived in [3] by a different approach. The last element (3, 3) of both G and H is equal to the value of γ corresponding to the resultant velocity given in equation (7). The RLR matrices D and E described in equations (5) and (6) also have γ_w as the (3, 3) element. Furthermore, all these matrices have 3 parameters. In the case of the LRL transformation, the 3 parameters are *u, θ,* and *v*. In the case of the RLR transformation, the 3 parameters are ϕ, *w,* and α. The matrices themselves have nine elements constructed from the 3 parameters.

Matrices G and D have to obey the following invariance for all space-time points (*x, y, t*) and (*x′, y′, t′*):

$$x^2 + y^2 - c^2t^2 = x'^2 + y'^2 - c^2t'^2 . \qquad (8)$$

Equation (8) places six constraints on the elements of G, H, D, and E. These constraints are obtained by equating the coefficients of x^2, y^2, t^2 on both sides and setting the coefficients of *xy, yt,* and *tx* as zero after expanding the right hand side of equation (8).

In particular, we will be utilizing two of these constraints, equations (9a) and (9b) below, for converting a G matrix to a D matrix.

$$g_{33}^2 - \frac{(g_{13}^2 + g_{23}^2)}{c^2} = 1;\ d_{33}^2 - \frac{(d_{13}^2 + d_{23}^2)}{c^2} = 1\ ; \qquad (9a)$$

$$g_{33}^2 - c^2(g_{31}^2 + g_{32}^2) = 1;\ d_{33}^2 - c^2(d_{31}^2 + d_{32}^2) = 1; \qquad (9b)$$

We show that given *u, θ,* and *v*, we can always generate an equivalent set of ϕ, *w*, and α such that G (*u* ,*θ* , *v*) is identical to D (ϕ, *w*, α); this procedure physically means the composition of the two planar boosts with velocities **u** and **v** resulting in a resultant velocity **w.**

However, we note that the reverse of this, i.e., conversion from D to G is possible only sometimes because in matrices G and D, the range of g_{22} is – 1 to +1 and the range of d_{22} is – γ to +γ.

3.2 Procedure for converting LRL (G–Matrix) to RLR (D–matrix)

In general, any LRL transformation (G matrix) created by the three parameters *u, θ,* and *v*, can be converted to an RLR transformation as follows.

We first set g_{33} = γ and obtain $(w/c) = \sqrt{1-(1/\gamma^2)}$ (it can be shown that $g_{33} \geq 1$ whenever $|u| \leq c$ and $|v| \leq c$).

We then express g_{13} and g_{23} as follows: $g_{13} = -w\gamma p$ and $g_{23} = w\gamma q$.

From the constraints dictated by equation (9a), it can be seen that $p^2 + q^2 = 1$. Therefore p and q can be expressed as cosα and sinα. Thus, by considering the last column of matrices G and D, we extract the parameter α for a given G matrix as

$$\alpha = \cos^{-1}\left(\frac{-g_{13}}{g_{33} * w}\right).$$

A similar procedure applied to the third row of G and D, along with equation (9b), gives us the parameter ϕ as

$$\phi = \cos^{-1}\left(\frac{-g_{31}}{g_{33} * (w/c^2)}\right).$$

The quadrant of the angles ϕ and α may be chosen appropriately by inspecting the sign of elements g_{32} (d_{32}) and g_{23} (d_{23}) respectively. So by considering the third row and third column of a G matrix with parameters *u, θ,* and *v*, we can always extract parameters ϕ*, w,* and α, of an equivalent D matrix; the other four elements of G and D, namely elements (1, 1), (1, 2), (2, 1), and (2, 2) are equal because of the constraints imposed by equation (8) on both G and D. The D matrix, when $|d_{22}| > 1$, cannot be converted to an equivalent G matrix. It is to be noted that the range of d_{22} is $-\gamma_w$ to $+\gamma_w$ and the range of g_{22} is – 1 to +1. Matrix D can be converted to an equivalent matrix G by a similar procedure, considering the second row and second column of D and G, only when $|d_{22}| \leq 1$. On the other hand, matrix G can always be converted to an equivalent matrix D. Thus D matrix constitutes a larger group that includes all G matrices.

For planar motion we have developed an Excel spreadsheet which automatically converts an LRL transformation to a RLR transformation and can be downloaded from [7].

4. Combining boosts in 3-dimensional space

In this section we extend our method to 3-dimensional space and show that two boosts in arbitrary directions in a 3-dimensional space compose into a single boost.

4.1 Representation of a boost in an arbitrary direction in 3-d space

A boost with a unit vector along its direction as represented in equation (10)

$$\cos\theta\,\mathbf{i} + \cos\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\sin\theta\,\mathbf{k} \tag{10}$$

can be visualized as subtending an angle *θ* with the *x*-axis and *(*90 – *θ)* with the *y-z* plane. To clarify further, we need to visualize the plane in which the *x*-axis and the unit vector along the boost under consideration are situated. This plane intersects the *y-z* plane along a line L. This line L, which is on the *y-z* plane and perpendicular to the *x*-axis subtends an angle *(*90 – *θ)* with the unit vector along the boost. Furthermore, L subtends an angle ϕ with the *y*-axis. Thus we arrive at equation (10) above for the unit vector along the boost in terms of the unit vectors along the coordinate axes namely **i**, **j**, and **k**.

4.2 Appropriate rotations for aligning the boost with the *x*-axis

By performing the following spatial rotations, we can align the *x*-axis along the boost.

- (1) Rotate the *y*-*z* plane by an angle ϕ
- (2) Rotate the *x*-*y* plane by an angle θ

Although the alignment of the *x*-axis and the line of a boost can be achieved by the more obvious combination of spatial rotations $\mathbf{R}_{xz}\mathbf{R}_{xy}$, we choose the alternative $\mathbf{R}_{xy}\mathbf{R}_{yz}$ by hindsight, to utilize the commutative property of $\mathbf{R}_{yz}$ and a boost along the *x*-axis. In the chosen $\mathbf{R}_{xy}\mathbf{R}_{yz}$ combination, the first rotation $\mathbf{R}_{yz}$ rotates the *yz* plane such that the *y*-axis is positioned in such a way that the *x*-axis, the line of the boost, and the *y*-axis, all lie on a plane. This enables the next rotation $\mathbf{R}_{xy}$ which rotates the (new) *xy* plane to align the *x*-axis with the line of the boost.

The two operations are to be performed in the order specified. The first operation is represented by the matrix:

$$\mathbf{R}_{yz\,(\Phi)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Similarly the second operation is represented by the matrix:

$$\mathbf{R}_{xy\,(\theta)} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The tip of the unit vector along the boost has the following coordinates:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ t \end{pmatrix},$$

where *t* is any instant in the inertial frame K. It is easy to verify that

$$\mathbf{R}_{xy\,(\theta)}\,\mathbf{R}_{yz\,(\Phi)} \begin{pmatrix} \cos\theta \\ \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ t \end{pmatrix}.$$

Thus we find that the two operations (done in the order indicated) $\mathbf{R}_{xy\,(\theta)}\,\mathbf{R}_{yz\,(\Phi)}$ align the *x*-axis with the direction of the boost as given in equation (10).

4.3 Composition of two boosts in 3-d space

For simplicity we assume that the *x*-axis is aligned with the first boost of magnitude *u*. To transport to the inertial frame M moving at velocity *u* along the *x*-axis with respect to inertial frame K, we perform the conventional Lorentz transformation $\mathbf{L}_{xu}$. The second boost of magnitude *v* may be in an arbitrary direction. So we first align the *x*-axis of M with the second boost as indicated in Section 4.2 and then perform a conventional Lorentz transformation of magnitude *v* to get

$$T = \mathbf{L}_{xv}\,\mathbf{R}_{xy\,(\theta)}\,\mathbf{R}_{yz\,(\Phi)}\,\mathbf{L}_{xu} \qquad (11)$$

The transformation T takes us to an inertial frame K' moving with a velocity *v* with respect to frame M along the direction of the second boost.

It can be easily shown by simple matrix multiplication (which in general is not commutative) that $\mathbf{R}_{yz\,(\Phi)}\,\mathbf{L}_{xu} = \mathbf{L}_{xu}\,\mathbf{R}_{yz\,(\Phi)}$.

The physical justification of this property of commutativity is because the transformation $\mathbf{L}_{xu}$ between *x* and *t* and the transformation $\mathbf{R}_{yz\,(\Phi)}$ between *y* and *z* are mutually exclusive and have no common parameter. Therefore, equation (11) can be rewritten as

$$T = \mathbf{L}_{xv}\,\mathbf{R}_{xy\,(\theta)}\,\mathbf{L}_{xu}\,\mathbf{R}_{yz\,(\Phi)}. \qquad (12)$$

The product $\mathbf{L}_{xv}\,\mathbf{R}_{xy\,(\theta)}\,\mathbf{L}_{xu}$ in equation (12) is on the *x-y* plane and we can use the procedure described in Section 3.2 to convert this LRL to RLR – that is, to $\mathbf{R}_{xy\,(A)}\,\mathbf{L}_{xw}\,\mathbf{R}_{xy\,(B)}$. Thus we obtain

$$T = \mathbf{R}_{xy\,(A)}\,\mathbf{L}_{xw}\,\mathbf{R}_{xy\,(B)}\,\mathbf{R}_{yz\,(\Phi)}\,. \qquad (13)$$

Equation (13) shows that two boosts of *u* and *v* in different directions in 3-dimensional space compose into a single boost of *w* with appropriate spatial rotations preceding and following the boost.

5. Conclusions

A formulation for combining two arbitrarily oriented boosts on a plane was developed. An Excel spreadsheet [7] can be downloaded for composing any two planar boosts. This method was extended to 3-dimensional motion by observing the property of commutativity of the rotation of the plane perpendicular to a boost and the transformation associated with that boost. Thus the combining of two boosts arbitrarily oriented in three spatial dimensions can be reduced to combining two boosts on a plane.

Thus we have shown that when we consider the fundamental building block as a uni-dimensional Lorentz transformation as specified in equation (2), the development of the Lorentz transformation in two- and three-dimensional spaces automatically ensures that two boosts always combine to yield a single boost. This property is inherent to the Lorentz transformation and does not pre-suppose the existence of a space-time continuum.

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