Journal of
Mathematics, Statistics and Allied Fields

Volume 1, Issue 2, 2007

# On the relationship between the Hausdorff dimension and the integration order

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**Abstract**

We use the Grünwald definition for the integration of the function $f : [0,1] \to \mathbb{R}$ to construct an iterated function system on the interval $[0,1]$. Then we show that this iterated function system satisfies the open set condition. Using this construction, we further show that the Hausdorff dimension is intimately related to the order of the fractional integration process for any real order $q > 0$.

## Introduction

In geometry, one often associates the notion of a dimension with a "measure", such as, the length of a line segment or the area of a region. Indeed, the usual integration of a function of one variable is described geometrically as the "area" under the curve. In this example, an integration applied once is said to be of *order* one, and clearly can be interpreted as corresponding to a region of (topological) dimension two. In [3], Grünwald extends the integration (or differentiation) of the function $f : [a,b] \to \mathbb{R}$ to non-integral real orders. In this paper, we show that the order $q > 0$ of the Grünwald integration process can be associated with a non-integral fractal dimension.

Let $[0,1] = C_0 \supset C_1 \supset C_2 \supset \dots$ be a decreasing sequence of sets, where $C_k$ is a finite union of disjoint closed intervals $C_k$ each interval of $C_k$ containing at least two intervals of $C_{k+1}$, and whose maximum length of the $k^{th}$ level intervals vanishes as $k \to \infty$. Then we form the totally disconnected subset of $[0,1]$, written $C = \cap_{k=0}^{\infty} C_k$, which we call the *generalized Cantor set* C. For our purposes, we define the generalized Cantor set as a collection of contractive maps $\{S_k(x)\}_{k=1}^{2}$ such that $C = \cup_{k=0}^{N} S_k(C)$ and $S_j \cap S_k = \emptyset$ for $j \neq k$, called an *iterated function system*. The contractive maps are defined by $S_k(x) = c_k x + d_k$, where $c_k, d_k \in \mathbb{R}$ and $c_k < 1$ for each $k$. The quintessential example of a "fractal" is Cantor's set $C$ with excluded middle region which is defined on the interval $[0,1]$ using the following collection of *contractive maps* $\{S_k(x)\}_{k=1}^{2}$.

$$S_1(x) = \frac{x}{3}$$
$$S_2(x) = \frac{x}{3} + \frac{2}{3}$$

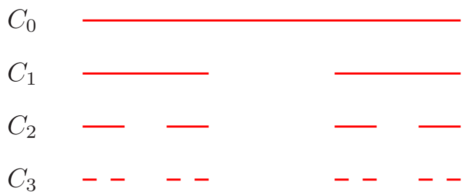


Figure 1: The first four iterations of Cantor's excluded middle set.

Let $C_0 = [0,1]$. The first iteration removes the middle third of the interval $[0,1]$, and is given by $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. The second iteration removes the middle ninth from the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$, and is given by $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{8}\right] \cup \left[\frac{8}{9}, 1\right]$. After the $n$-th iteration, $C_n$ is composed of $2^n$ sub-intervals each of width $3^{-n}$. Since each iteration of open intervals is contained in the previous construction, we say that Cantor's set satisfies the *open set condition.* In the limit, we obtain Cantor's set whose (similarity or Hausdorff) dimension is given by the following computation.

$$s = \frac{log 2^n}{log 3^{-n}} = 0.6309\ldots$$

Using Cantor's set as a guide, let us now consider Riemann-Darboux definition for the integration of a function $f : [a,b] \to \mathbb{R}$ over an interval $[a,b]$. We divide the interval $[a,b]$ using a finite collection of points $\{x_{N-k}\}_{k=0}^{N}$, called the *partition points*, such that $a = x_0 < x_1 < \cdots < x_N = b$. The *width* $x_k - x_{k-1}$ of each sub-interval $[x_{k-1}, x_k]$ is denoted by $\Delta x_k$. From each sub-interval, we select a point $x_k^*$, which we call a *sample point*. We associate a rectangle of height $f(x_k^*)$ and width $\Delta x_k$ to each sub-interval. The "area" under the curve is approximated by forming a Riemann-Darboux sum of the rectangular areas associated to each sub-interval. We further divide each sub-interval, called a *refinement*, so that as the length of sub-intervals tends to zero, we obtain the Riemann-Darboux integral

$$\int_a^b f(x)dx = \lim_{N\to\infty} \left\{ \sum_{k=0}^{N-1} f\left(x_{N-k}^*\right) \Delta x_{N-k} \right\},$$

when the limit exists. In Figure 2, we compare the upper Darboux sum of the function $f : [0,1] \to \mathbb{R}$ given by $f(x) = x^2$ when the number of subdivisions $N$ is 3 and 9. Note that if we started with three sub-intervals, then we can make further refinements by dividing each sub-interval into thirds to obtain nine sub-intervals. We can, of course, continue this refinement recursively in this manner to obtain $3^n$ each sub-interval of width $3^{-n}$, where $n$ is the number of iterations. We remark that the procedure of this refinement is similar to the construction of Cantor's set. In general, we can use any partition and subsequent refinements to construct the Darboux sum since these representations are equivalent in the limit. We remind the reader that integration of order one corresponds to the area of a two-dimensional region.

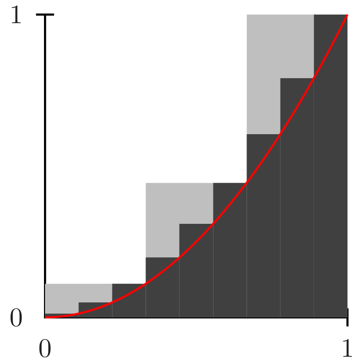


Figure 2: The upper Darboux sum of $f(x) = x^2$ for $N = 3$ (light gray) and $N = 9$ (dark gray).

Applying the Riemann-Darboux definition $m$-times, the $m$-fold integration, denoted by $D_x^{-m} f(x)$, of the function $f : [a, b] \to \mathbb{R}$ over an interval $[a, b]$ is given by

$$D_x^{-m} f(x) = \lim_{N\to\infty} \left\{ \sum_{k=0}^{N-1} \binom{k+m-1}{k} f\left(x^*_{N-k}\right) \Delta^m x_{N-k} \right\},$$

where $m$ is the *integration order*, $\Delta^m x_{N-k}$ is the product of $\Delta x_{N-k}$ $m$-times and the binomial coefficients are defined by $\binom{i}{j} = \frac{i!}{j!(i-j)!}$ for $i, j \in \mathbb{N}$. We note that the correct contribution from each term in the sum is provided by the binomial coefficients. To construct the Riemann-Darboux integral, we again divide the interval $[a, b]$ using a finite collection of sub-intervals of width $\Delta x_k$. Selecting a sample point $x^*_k$ from each sub-interval, we associate a rectangle of height $\binom{k+m-1}{k} f(x^*_k)$ and width $\Delta^m x_k$ to each sub-interval. The (generalized) "area" under the curve is approximated by forming an $m$-fold Riemann-Darboux sum of the rectangular areas associated to each sub-interval. As the length of sub-intervals tends to zero, we obtain the $m$-fold Riemann-Darboux integral of the function $f(x)$.

Figure 3 compares two representations of the Riemann-Darboux sum of the function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x^2$ when the integration order is two. In both Figure 3 (a) and (b), we divide the interval $[0, 1]$ into sub-intervals by the same procedure used in Figure 2. Note that in Figure 3 (a) we have shifted the three dark gray rectangles to fit inside the first light gray rectangle for each sub-interval of width $\Delta x_k$ to obtain Figure 3 (b). We remark again that the procedure for refinement is similar to the construction of Cantor's set.

More generally, Grünwald ([3]) defines the process of integration or differentiation, denoted by $D_x^{-q} f(x)$, to any order $q \in \mathbb{R}$ with respect to $x$ of the function $f : [a, b] \to \mathbb{R}$. We can now consider Grünwald's definition for integration of a function to the order $q > 0$. We divide the interval $[a, b]$ into sub-intervals $[x_{k-1}, x_k]$ and select a sample point $x^*_k$. Let $\Delta^q x_k$ be the $q^{th}$-root of $\Delta x_k$. Then the $k^{th}$ rectangle is of width $\Delta x^q_{N-k}$ and height $\frac{1}{\Gamma(q)} \frac{\Gamma(k+q)}{\Gamma(k+1)} f(x^*_{N-k})$. We form a Riemann-Darboux sum of the rectangular areas

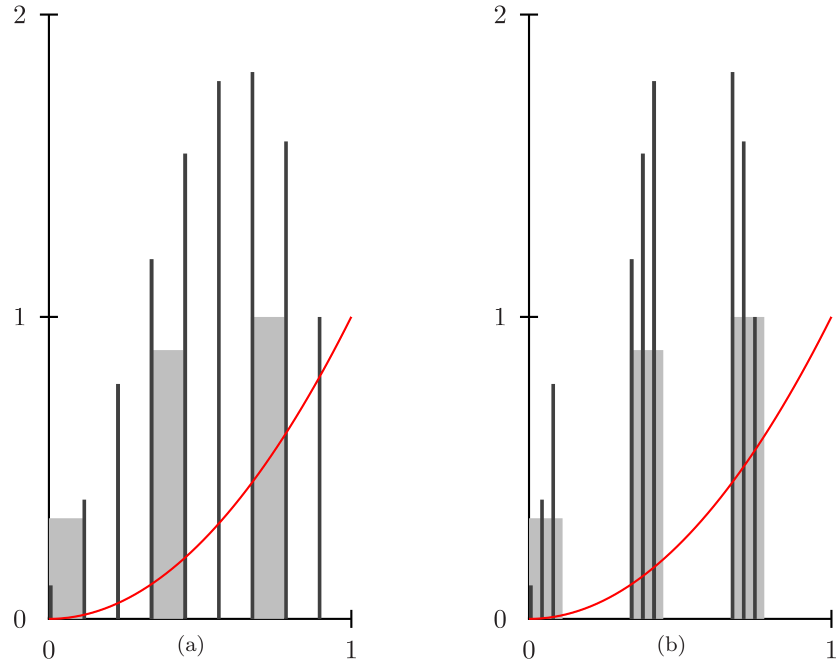


Figure 3: (a) The Riemann-Darboux sum of $f(x) = x^2$ for $N = 3$ (light gray) and $N = 9$ (dark gray) when $m = 2$. (b) An equivalent Riemann-Darboux sum representation.

associated to each sub-interval and obtain the fractional integral

$$D_x^{-q} f(x) = \lim_{N\to\infty} \left\{ \frac{1}{\Gamma(q)} \sum_{k=0}^{N-1} \frac{\Gamma(k+q)}{\Gamma(k+1)} f\left(x_{N-k}^*\right) \Delta^q x_{N-k} \right\},$$

where we denote the gamma function at $x \in \mathbb{R}$ by $\Gamma(x)$. The Grünwald integral represents the "generalized" area under the curve.
After presenting some key definitions and results, we construct an iterated function system using Grünwald's definition for the integration of a function $f : [0,1] \to \mathbb{R}$ on the interval $[0,1]$. We also show that this iterated function system satisfies the open set condition. Finally, we associate the order of the integration process to the (Hausdorff) dimension.

## Preliminaries

We present basic definitions and results on fractional calculus from Grünwald ([3]) and Oldham and Spannier ([6]); on fractals from Edgar ([1]), Falconer ([2]) and Hutchinson ([5]); and on general measure theory from Halmos ([4]) and Mattila ([7]).

Definition 1. A (finite) *partition* $\pi$ of the interval $[a,b]$ is a finite collection of points $\{x_0, x_1, \ldots, x_{N-1}\}$, called partition points, such that $a = x_0 < x_1, \ldots, < x_{N-1} = b$. The length of the interval $[x_{j-1}, x_j]$ is denoted by $\Delta x_j = x_j - x_{j-1}$. The collection of all (finite) partitions of the interval $[a,b]$ is denoted by $\Pi[a,b]$.

Definition 2. A partition $\pi'$ is called a *refinement* of the partition $\pi$ if every partition point of $x_j \in \pi$ also belongs to $\pi'$.

Remark 1. The process of integration or differentiation, denoted by $D_x^q f(x)$, to any order $q \in \mathbb{R}$ with respect to $x$ of the function $f : [a, b] \to \mathbb{R}$ is given in [2] by Grünwald. If $q < 0$, $q = 0$ or $q > 0$ then we say that the process is a differentiation, the identity map or an integration, respectively. When $q = 1$, the Grünwald's definition reduces to the ordinary integral of Riemann.

Definition 3. (*Grünwald definition*) Let $f : [a, b] \to \mathbb{R}$ is a continuous function on the interval closed $[a, b] \subset \mathbb{R}$. The derivative or integration to order $q$ is given by

$$D_x^{-q} f(x) = \lim_{N\to\infty} \left\{ \frac{1}{\Gamma(q)} \sum_{k=0}^{N-1} \frac{\Gamma(k+q)}{\Gamma(k+1)} f\left(x^*_{N-k}\right) \Delta^q x_{N-k} \right\},$$

where we denote the gamma function at $x \in \mathbb{R}$ by $\Gamma(x)$.

Definition 4. A map $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a *contractive map* if there exist an $r$, called the *contraction ratio* with $0 < r < 1$, such that for all $x, y \in X$, $|S(x) - S(y)| = r|x - y|$.

Definition 5. Let $I$ be an index set, possibly infinite. The collection of contractive maps $\{S_k : X \to X | k \in I\}$ on a closed interval $X \subset \mathbb{R}$ is called an *iterated function system* or IFS.

Definition 6. The IFS $\{S_k\}$ is said to satisfy the *open set condition* iff there exists a nonempty open set $U$ for which we have $S_i(U) \cap S_j(U) = \emptyset$ for $i \neq j$ and $U \supseteq S_i(U)$ for all $i$.

Definition 7. Let $\{S_k\}$ be an IFS. We denote a list of contraction ratios by $(r_1, r_2, \ldots, r_N)$. If $\sum_k r_k^s = 1$ then we call $s$ the *similarity dimension* of the IFS.

## The Grünwald Definition and Iterated Function Systems

In this section, we define a partition and subsequent refinements of Grünwald's definition for integration as an iterated function system on the interval $[0, 1]$. Also, we show that it satisfies the open set condition for integration order $q > 0$.

Definition 8. Let $N \geq 2$. Then for $q \geq 1$, the collection of maps $\{S_k(x)\}_{k=1}^N$ given by

$$S_k(x) = \frac{x}{N^q} + \frac{k-1}{N}$$

is called the *Grünwald IFS, which we denote by $\mathcal{G}$.*

Theorem 1. *If $q > 0$ then the Grünwald IFS $\mathcal{G}$ satisfies the open set condition.*

P r o o f. Let $N \geq 2$, $q \geq 1$ and $\mathcal{G}_0 = [0, 1]$. We denote the $n^{th}$ iteration of the interval $[0, 1]$ by $\mathcal{G}_n$. We show by induction on $n$. Suppose $n = 1$. Then for any $k$, the contraction $S_k(x)$ sends the interval $(0, 1)$ to the interval $\left(\frac{k-1}{N}, \frac{1}{N^q} + \frac{k-1}{N}\right)$. We observe that the previous and next contractive maps, $S_{k-1}(x)$ and $S_{k+1}(x)$, send the interval $(0, 1)$ to the intervals $\left(\frac{k-2}{N}, \frac{1}{N^q} + \frac{k-2}{N}\right)$ and $\left(\frac{k}{N}, \frac{1}{N^q} + \frac{k}{N}\right)$, which are to the left and right of the interval $\left(\frac{k-1}{N}, \frac{1}{N^q} + \frac{k-1}{N}\right)$. Clearly, these intervals do not overlap any other interval, and they are contained in $(0,1)$.

Now, assume that the Grünwald IFS satisfies the open set condition at $n = j$. Applying the contractive map

$$S_k^j(x) = \frac{x}{N^{jq}} + \sum_{m=0}^{j} \frac{k-1}{N^{1+mq}}$$

to the interval $(0,1)$ $j$-times, we obtain the interval $\left(\sum_{m=0}^{j} \frac{k-1}{N^{1+mq}}, \frac{1}{N^{jq}} + \sum_{m=0}^{j} \frac{k-1}{N^{1+mq}}\right)$. Again, we observe that the previous and next contractive maps, $S_{k-1}(x)$ and $S_{k+1}(x)$, sends the interval $(0,1)$ to the intervals $\left(\sum_{m=0}^{j} \frac{k-2}{N^{1+mq}}, \frac{1}{N^{jq}} + \sum_{m=0}^{j} \frac{k-2}{N^{1+mq}}\right)$ and $\left(\sum_{m=0}^{j} \frac{k}{N^{1+mq}}, \frac{1}{N^{jq}} + \sum_{m=0}^{j} \frac{k}{N^{1+mq}}\right)$, which are to the left and right of the interval $\left(\sum_{m=0}^{j} \frac{k-1}{N^{1+mq}}, \frac{1}{N^{jq}} + \sum_{m=0}^{j} \frac{k-1}{N^{1+mq}}\right)$. These intervals again do not overlap any other intervals, and are contained in $(0,1)$. The inductive step

$$S_k \circ S_k^j(x) = \frac{1}{N^q}\left(\frac{x}{N^{jq}} + \sum_{m=0}^{j} \frac{k-1}{N^{1+mq}}\right) + \frac{k-1}{N} = S_k^{j+1}(x)$$

provides the conclusion for $n \geq 1$.

Let $0 < q < 1$. We observe that the intervals overlap under the application of the contractive maps $\{S_k(x)\}_{k=1}^{N}$. However, using the Grünwald definition we can always adjust the length of the intervals thereby satisfying the open set condition.

**Main Results**

Since the Grünwald IFS $\mathcal{G}$ satisfies the open set condition, then the Hausdorff dimension is equal to the similarity dimension ([1], [5], [7]). We compute the Hausdorff dimension for the Grünwald's IFS $\mathcal{G}$. Then we associate the Hausdorff dimension to the Grünwald's definition of fractional integration.

THEOREM 2. *The Hausdorff dimension of the Grünwald IFS $\mathcal{G}$ is $s = \frac{1}{q}$.*

P r o o f. At the $n^{th}$ iteration, $\mathcal{G}_n$ can be covered by $N^n$ intervals of length $N^{-nq}$. Thus, the similarity dimension is required to satisfy the following condition.

$$\sum_k r_k^s = N^n N^{-nsq} = 1$$

Note that the Hausdorff dimension for any interval $[a,b]$ of $\mathbb{R}$ is one ([1]). In [2], the dimension of the product space $A \times B$ is $dimA + dimB$. Since the Grünwald definition for the integration process is defined on a product space, we have the following theorem.

THEOREM 3. *The Hausdorff dimension of $\mathcal{G} \times \mathbb{R}$ is $s = 1 + \frac{1}{q}$.*

## Future Directions

In this paper, we have developed a relationship between integration of order $q$ for functions of one variable and the dimension $s$ of the domain. Now, we suggest some possible directions for this work, which we plan to investigate in the future.

1. Let $\mathbb{R}^n$ be the Cartesian product of $n$ copies of $\mathbb{R}$. Then, for functions of many variables $f : \mathbb{R}^{n-1} \to \mathbb{R}$ whose domain has the fractal character of the $(n-1)$-product of the Cantor set, we expect the following relationship between integration order and Hausdorff dimension for $\mathcal{G} \times \cdots \times \mathcal{G} \times \mathbb{R}$ with no overlap.

$$s = 1 + \sum_{i=1}^{n-1} \frac{1}{q_i}$$

   How does this relationship change if we allow overlap? How do these results extend to the complex pane? If the integration order is complex, then what does a complex dimension mean?

2. Is it possible for two integration processes of order $p$ and $q$ to occur at once? Can a single integration process be decomposed into several integration processes?

3. The concept of entropy is well known in the study of fractals. Can we associate the concept of entropy to an integration process of order $q$ for functions of one variable? How does the concept of entropy extend to higher dimensions? Are there any applications to image or information processing?

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