Journal of
Mathematical Sciences ISSN 1934-7286
Volume 1, Issue 1, 2007

Some remarks concerning the representation of certain subnormal derivations and their Hilbert-Schmidt norm estimate

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Abstract

In this note we extend a result of D. Jocic concerning the representation of some derivations. Precisely, we prove that if A and B^* are subnormal operators on a Hilbert space H and $X \in L(H)$ is such that $AX - XB \in C_2(H)$, then $f(A)X - Xf(B)$ and $f(A)^*X - Xf(B)^*$ belong to $C_2(H)$ and each of these expressions has a certain integral representation when the function f belongs to the class $Lip\,(\Sigma) \cap R(\Sigma)$, where $\Sigma = \sigma(A) \cup \sigma(B)$. Furthermore, Hilbert-Schmidt norm estimates of $f(A)X - Xf(B)$ and $f(A)^*X - Xf(B)^*$ are given.

1. Introduction.

The structure of generalized derivations has been studied when operators A, B belong to various classes, such as normal operators, subnormal operators, hyponormal operators. On one hand, an area of interest in generalized derivations is whether $\delta_{f(A),f(B)}(X) := f(A)X - Xf(B)$ remains in a certain ideal, such as the ideal of compact operators or some Shatten class, C_p, when $\delta_{A,B}(X) := AX - XB$ is in the same ideal, (cf. [1], [4], [5], [6], [9]). On the other hand, the orthogonality of the $Ker(\delta_{A,B})$ and $Ran(\delta_{A,B})$ has been investigated (cf. [1], [2], [3], [7], [8], [10], [11]). In his paper [6] D. Jocic developed a theory concerning the integral representation of derivations of the form $\delta_{f(A),f(B)}$ when operators A, $B \in L(H)$ are normal, $f \in Lip\,(\sigma(A) \cup \sigma(B))$, and $X \in L(H)$ satisfies $AX - XB \in C_2(H)$. In this note, we make use of such representation and extend the result to subnormal derivations and obtain some Hilbert-Schmidt (HS) norm estimates of the operator $f(A)X - Xf(B)$ in terms of the HS-norm of $AX - XB$. In [9], the author obtained HS-norm estimates for $f(A)X - Xf(B)$ in the case in which operators A, B are hyponormal and have some additional spectral properties and f belongs to a proper subalgebra of $Lip\,(\Sigma)$. Since subnormal operators are hyponormal operators, we recapture the estimates obtained in [9], but in a more general context concerning subnormal operators. Furthermore, the HS-norm estimates are obtained as a consequence of the integral representation of subnormal derivations.

2. Preliminaries.

Let H be a separable, infinite dimensional, complex Hilbert space, and denote by $L(H)$ the algebra of all bounded linear operators on H, by $C_2(H)$ the Hilbert-Schmidt class, by $C_1(H)$ the trace-class and by $\mathrm{tr}(T)$ the complex-valued *trace* of an operator $T \in C_1(H)$. The class of Hilbert-Schmidt operators is a Hilbert space with the norm $||\cdot||_2$ which arises from the inner product $(X, Y) = \mathrm{tr}(XY^*) = \mathrm{tr}(Y^*X)$. For $T \in L(H)$, $\sigma(T)$ denotes the spectrum of T, and for a compact subset $\Sigma \subset \mathbb{C}$, $Lip\,(\Sigma)$ denotes the set of Lipschitz functions on Σ. Furthermore, $Rat(\Sigma)$ denotes the algebra of rational functions with poles off Σ, and $R(\Sigma)$ denotes the the closure of $Rat(\Sigma)$ in the supremum norm over Σ. Recall that an operator T is normal (hyponormal) if $T^*T - TT^* = 0$ ($T^*T - TT^* \geq 0$), respectively. The operator T is subnormal if T is the restriction of a normal operator to a nontrivial invariant subspace $M \subseteq H$, that is, $TM \subseteq M$.

For fixed operators $A, B \in L(H)$, the mapping $\delta_{A,B} : L(H) \to L(H)$ defined by $\delta_{A,B}(X) = AX - XB$ is called a generalized derivation. If A, B are normal operators, then

$\delta_{A,B}$ is called a normal derivation. If A, B^* are subnormal (hyponormal) operators, then we will call $\delta_{A,B}$ a subnormal (hyponormal) derivation, respectively. In this note, we will not be concerned with the class of hyponormal derivations, but only subnormal derivations.

We briefly summarize from [6] the steps that lead to such a representation. For the normal operators A, $B \in L(H)$, let E, F be their spectral measures, respectively. For X, $Y \in C_2(H)$, define $\mu_{X,Y}$ on Borel rectangles $\gamma \times \delta$ in $\mathbb{C}^2$ by $\mu_{X,Y}(\gamma \times \delta) = \mathrm{tr}(E(\gamma)XF(\delta)Y^*)$. According to Theorem 2.1 of [6] (see also [5]), $\mu_{X,Y}$ has a unique extension to a complex Borel measure on $\mathbb{C}^2$ which satisfies $|\,\mu_{X,Y}|(\mathbb{C}^2) \leq ||X||_2\,||Y||_2$.

Since for $f \in L^\infty(\sigma(A) \times \sigma(B), \mu_{X,Y})$,

$$|\int_{\sigma(A)\times\,\sigma(B)} f(z,w)\,d\mu_{X,Y}| \leq ||f||_\infty\,|\mu_{X,Y}(\mathbb{C}^2)| \leq ||f||_\infty\,||X||_2||Y||_2,$$

the mapping $(X,Y) \mapsto \int_{\sigma(A)\times\,\sigma(B)} f(z,w)\,d\mu_{X,Y}$ is a bounded sesqui-linear functional on $C_2(H) \times C_2(H)$. Therefore, there exists a unique bounded linear operator

$$f(A,B) \in L(C_2(H))$$

such that

$$\mathrm{tr}[(f(A,B)X)Y^*] = \int_{\sigma(A)\times\,\sigma(B)} f(z,w)\,d\mu_{X,Y},$$

and $||f(A,B)|| \leq ||f||_\infty$. In particular,

$$f(A,B)X \in C_2(H) \text{ and } ||f(A,B)X||_2 \leq ||f||_\infty ||X||_2,$$

whenever $X \in C_2(H)$,

Theorem A ([6]). *If A, $B \in L(H)$ are normal operators and $X \in L(H)$ satisfies $AX - XB \in C_2(H)$, and if $f \in Lip\,(\sigma(A) \cup \sigma(B))$, then $f(A)X - Xf(B) \in C_2(H)$ and $f(A)X - Xf(B) = \check{f}(A,B)(AX - XB)$, where*

$$\check{f}(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w}, & z \neq w \\ 0, & z = w. \end{cases}$$

We will make use of the following extension of Fuglede-Putnam theorem for subnormal operators.

Theorem B ([4]). *If A, $B^* \in L(H)$ are subnormal operators and $X \in L(H)$ is such that $AX - XB \in C_2(H)$, then $A^*X - XB^* \in C_2(H)$ and*

$$||A^*X - XB^*||_2 \leq ||AX - XB||_2.$$

3. Some subnormal derivations.

In this section we investigate the validity of such representation when A, B^* are subnormal operators.

The following lemma is necessary.

Lemma 1 *If* $S_1, S_2 \in L(H)$ *are subnormal operators, then there exist a Hilbert space* $K \supseteq H$ *and some normal operators* $N_1, N_2 \in L(K)$ *which are extensions of* S_1, S_2*, respectively, and* $\sigma(N_i) \subseteq \sigma(S_i),\ i = 1, 2.$

Proof. It is well known that each subnormal operator has a minimal normal extension that is unique up to unitary equivalence and the spectrum of the minimal normal extension is a subset of the spectrum of the subnormal operator. Thus, for each $S_i,\ i = 1, 2$, there are Hilbet spaces $H_i,\ i = 1, 2$, and normal operators $N'_i \in L(H \oplus H_i)$ with $\sigma(N'_i) \subseteq \sigma(S_i)$ having the following matrix representation:

$$N'_i = \begin{pmatrix} S_i & A_i \\ 0 & B_i \end{pmatrix}.$$

Let $H' = H_1 \oplus H_2$ and $\lambda_i \in \sigma(N'_i)$ be arbitrary points and let $N_i \in L(H \oplus H')$ have the following matrix representation with respect to $H \oplus H_1 \oplus H_2$:

$$N_1 = \begin{pmatrix} S_1 & A_1 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & \lambda_1 I_{H_2} \end{pmatrix}$$

and

$$N_2 = \begin{pmatrix} S_2 & 0 & A_2 \\ 0 & \lambda_2 I_{H_1} & 0 \\ 0 & 0 & B_2 \end{pmatrix}.$$

Operators $N_1,\ N_2$ satisfy the desired properties.

For a subnormal operator $S \in L(H)$ and a function $f \in R(\sigma(S))$, one can associate an operator $f(S) \in L(H)$ as follows. Let $r_n \in Rat(\sigma(S)),\ n \in \mathbb{N}$, such that $||f - r_n||_{\sigma(S),\infty} \to 0$, as $n \to \infty$, and let $N_S \in L(K)$, where $K \supset H$, be the minimal normal extension of S. Since $\sigma(N_S) \subseteq \sigma(S)$, we have

$$r_n(N_S) = \begin{pmatrix} r_n(S) & S'_{12} \\ 0 & S'_{22} \end{pmatrix},$$

and $r_n(N_S) \to f(N_S)$ in the operator norm of $L(K)$. Therefore $r_n(S)$ converges in the operator norm of $L(H)$ to an operator that will be denoted by $f(S)$. It is obvious that this operator does not depend on the sequence $\{r_n\}$. In a similar way, for $f \in R(\sigma(T))$, one can define $f(T)$, when $T^* \in L(H)$ is a subnormal operator.

Theorem 2. *Let* $A, B^* \in L(H)$ *be subnormal operators and* $X \in L(H)$ *such that* $AX - XB \in C_2(H)$*, and let* $f \in Lip\,(\Sigma) \cap R(\Sigma)$*, where* $\Sigma = \sigma(A) \cup \sigma(B)$*. Then there exists a Hilbert space* $K \supseteq H$ *such that*

$$(f(A)X - Xf(B)) \oplus 0 = [\check{f}(N_A,\, N^*_{B^*})((AX - XB) \oplus 0)],$$

where $N_A, N_{B^*} \in L(K)$ *are normal extensions of* A, B^* *as in Lemma 1. Therefore,*

$$||f(A)X - Xf(B)||_2 \le ||\check{f}||_\infty \cdot ||AX - XB||_2.$$

Proof. Let A, B^* be subnormal operators in $L(H)$, and let K and $N_A, N_{B^*} \in L(K)$ with $\sigma(N_A) \subseteq \sigma(A),\ \sigma(N_{B^*}) \subseteq \sigma(B^*)$, be their normal extensions according to Lemma 1. Relative to the decomposition $K = H \oplus H^\perp$, we write

$$N_A = \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad N_{B^*} = \begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix}.$$

If we put $\tilde{X} = X \oplus 0$ on $H \oplus H^{\perp}$, then we have $N_A\,\tilde{X} - \tilde{X}\,N^*_{B^*} = (AX - XB) \oplus 0$, and therefore $N_A\,\tilde{X} - \tilde{X}\,N^*_{B^*} \in C_2(K)$.

If $r \in Rat(\Sigma)$, where $\Sigma = \sigma(A) \cup \sigma(B)$, then one can see that

$$r(N_A) = \begin{pmatrix} r(A) & A'_{12} \\ 0 & A'_{22} \end{pmatrix} \text{ and } r(N^*_{B^*}) = \begin{pmatrix} r(B) & 0 \\ B'_{21} & B'_{22} \end{pmatrix}.$$

Thus, if $f \in Lip\,(\Sigma) \cap R(\Sigma)$, using a limiting argument, one can say that $f(N_A)$ and $f(N^*_{B^*})$ have similar matrix representation to those in (1), but with f replacing r. According to Theorem A,

$$f(N_A)\tilde{X} - \tilde{X}f(N^*_{B^*}) \in C_2(K)$$

and

$$f(N_A)\tilde{X} - \tilde{X}f(N^*_{B^*}) = \check{f}(N_A,\, N^*_{B^*})(N_A\tilde{X} - \tilde{X}N^*_{B^*}).$$

Since $f(N_A)\tilde{X} - \tilde{X}f(N^*_{B^*}) = (f(A)X - Xf(B)) \oplus 0$, the proof is finished.

For the Hilbert spaces H and K in Theorem 2, let P denote the orthogonal projection of K onto H.

Corollary 3. *Let $A,\, B^* \in L(H)$ be subnormal operators and $X \in L(H)$ such that $AX - XB \in C_2(H)$, and let $\Sigma = \sigma(A) \cup \sigma(B)$ and $f \in Lip\,(\Sigma) \cap R(\Sigma)$. Then*

$$f(A)^*X - Xf(B)^* = P[\check{g}(N^*_A, N_{B^*})(N^*_A\tilde{X} - \tilde{X}N_{B^*})]P,$$

where $g(w) = \overline{f(\overline{w})}$. Consequently, $f(A)^*X - Xf(B)^* \in C_2(H)$ and

$$||f(A)^*X - Xf(B)^*||_2 \le ||\check{f}||_\infty \cdot ||AX - XB||_2.$$

Proof. Since $N_A\tilde{X} - \tilde{X}N^*_{B^*} = (AX - XB) \oplus 0 \in C_2(K)$, according to Fuglede-Putnam theorem (cf. [4]), $N^*_A\tilde{X} - \tilde{X}N_{B^*} \in C_2(K)$ and

$$||\,N_A\tilde{X} - \tilde{X}N^*_{B^*}||_2 = ||\,N^*_A\tilde{X} - \tilde{X}N_{B^*}||_2.$$

Therefore, according to Theorem A,

$$g(N^*_A)\tilde{X} - \tilde{X}g(N_{B^*}) = \check{g}(N^*_A, N_{B^*})(N^*_A\tilde{X} - \tilde{X}N_{B^*}) \in C_2(K).$$

On the other hand,

$$g(N^*_A) = f(N_A)^* = \begin{pmatrix} f(A)^* & 0 \\ A''_{12} & A''_{22} \end{pmatrix}$$

and

$$g(N_{B^*}) = f(N^*_{B^*})^* = \begin{pmatrix} f(B)^* & B''_{12} \\ 0 & B''_{22} \end{pmatrix}.$$

Thus

$$g(N^*_A)\tilde{X} - \tilde{X}g(N_{B^*}) = \begin{pmatrix} f(A)^*X - Xf(B)^* & \star \\ \star & \star \end{pmatrix},$$

which, after taking the compression of the above operator on the space H, concludes the first part of the corollary. The second part is a consequence of the first part and Fuglede-Putnam's theorem for subnormal operators [4].

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