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**The Inverse Boundary Value Problem for the Two-Dimensional Elliptic Equation in Anisotropic Media**

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**Abstract**

In this article we consider the inverse boundary value problem for the elliptic equation $\nabla \cdot (\gamma(x)\nabla u) = 0$ in $\mathbb{R}^2$. We use Fadeev's fundamental solution and a method previously employed by Sylvester, which consists of reducing the anisotropic conductivity $\gamma$, to an isotropic one. This results, after a change of dependent variables, in a Schrödinger equation with the potential $q(x)$. Then using Nachman and the $\overline{\partial}$-equation, we give a different proof of his result, that the Dirichlet-to-Neumann map $\Lambda$ determines the coefficients $\gamma$ of the equation uniquely, up to a change of coordinates.

# 1 Introduction

Inverse problems arise naturally in the physical world around us. The inverse boundary value problem consists of gaining some knowledge about the interior of a body from measurements on the boundary. In medical diagnostics, for example, one is interested in determining the location and/or the size of a tumor inside the body from measurements taken just on the outside. There are also applications in the earth sciences, for example, where one uses measurements taken on the surface in order to locate oil or minerals inside earth.

We are interested in uniqueness results for the inverse problems. To state our inverse boundary value problem, first consider the following boundary value problem

$$\begin{aligned} \nabla \cdot (\gamma(x)\nabla u) &= 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} &= f \end{aligned} \tag{1.1}$$

where $\Omega$ is some bounded subset of $\mathbb{R}^2$, $\partial\Omega$ is $C^\infty$, $f \in C^\infty(\partial\Omega)$ and $\gamma(x)$ is a positive definite symmetric matrix. If (1.1) has a unique solution for each $f$, we can define a Dirichlet-to-Neumann operator

$$\begin{aligned} \Lambda &: \partial\Omega \to \partial\Omega \quad \text{by} \\ \Lambda f &= \frac{\partial u}{\partial n}\Big|_{\partial\Omega}, \end{aligned} \tag{1.2}$$

where $n$ is the exterior unit normal vector to $\partial\Omega$.

The inverse boundary value problem for (1.1) consists of determining $\gamma(x)$ from the knowledge of $\Lambda$. Notice that this problem is equivalent to the impedance tomography problem, where the goal is to use voltage and current measurements on the boundary in order to find the conductivity of a given material. Here $\gamma(x)$ represents the electrical conductivity of some material, $u(x)\big|_{\partial\Omega}$ is the voltage and $\frac{\partial u}{\partial n}\Big|_{\partial\Omega}$ is the current on the boundary. So our Dirichlet-to-Neumann map is a set of values of $u(x)\big|_{\partial\Omega}$ and corresponding $\frac{\partial u}{\partial n}\big|_{\partial\Omega}$, and is often called the "voltage-to-current" map.

To answer wether $\Lambda$ determines $\gamma(x)$ uniquely, which is the question that was posed by Calderon [5], there has been a body of work produced in the last few decades. Kohn and Vogelius [16] proved that $\Lambda$ determines $\gamma(x)$ and all of its derivatives, but only on the boundary, if $\partial\Omega$ is $C^\infty$. Sylvester and Uhlmann [24] proved for $n \geq 3$ that $\Lambda$ uniquely determines $\gamma$, if $\partial\Omega$ is $C^\infty$ and $\gamma$ is in $C^\infty(\overline{\Omega})$ (global uniqueness). The 2-dimensional case, however, is the hardest, as unlike in $n \geq 3$, the inverse problem for $n = 2$ is not overdetermined. For $n = 2$, Sylvester and Uhlmann [23] showed uniqueness (up to a change of coordinates) for $\gamma(x)$, when $\log(\det\gamma(x))$ is small in $C^3$. This result was strengthened by Nachman [18] who showed that for $n = 2$, $\Lambda$ determines $\gamma(x)$ uniquely (up to a change of coordinates) without the extra assumption used in [23]. Then a new proof of the main result of [18] was provided by Brown and Uhlmann [4], where instead of considering a second order Schrödinger equation, they studied scattering for a particular first order system, as was previously done by Beals and Coifman [3]. Other uniqueness results in two dimensions were obtained by Grinevich and Novikov [13], by Isakov and Nachman [14], by Isakov and Sun [15], by Mandache [17], with one of the more recent ones by Astala, Lassas and Päivärinta [2].

In this paper we provide a different proof of Nachman's result. We believe that the methods used in our proof (e.g. the absence of the essential points discussed in Section 6) can be used to solve the inverse boundary value problem for a more general second order operator.

We shall prove the following theorem

**Theorem 1** *Let*

$$L_p(x, -i\partial_x) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(\gamma_p^{ij}\frac{\partial}{\partial x_j}),\ p = 1, 2 \tag{1.3}$$

*with corresponding $\Lambda_p$ defined as in (1.2).*
*If $\Lambda_1 = \Lambda_2$, then there exists a diffeomorphism $y = S(x)$, such that*

$$\gamma_2(y) = \frac{(J_S(x))^T \gamma_1(x) J_S(x)}{det(J_S(x))}$$

*where $J_S(x)$ is the Jacobian matrix of $y = S(x)$.*
*Moreover,*

$$S = I \text{ on } \partial\Omega.$$

To show that $\Lambda$ determines $\gamma(x)$ uniquely (up to a change of coordinates), we proceed as follows. In Section 2, we use Fadeev's fundamental solution and introduce two intermediate objects $h_0(x,t)$ and $h_1(x,t)$. We then show that $\Lambda$ determines $\widetilde{h_1}(-2z_R(t),t)$ uniquely (where $\widetilde{h_1}$ stands for the Fourier transform of $h_1$). Note that our proof here is similar to that of Nachman except for the fact that we first consider our operator $L$ as a sum of the negative Laplacian $-\triangle$ and $M$, where $M$ is a differential operator with compact support, while Nachman considers $L$ as a sum of $-\triangle$ and $q$, where $q$ is just a function with compact support. In Section 3, we follow a method previously employed by Sylvester [21], and by Eskin and Ralston [11], which consists of reducing the anisotropic conductivity $\gamma$, to an isotropic one, which results, after a change of dependent variables in a Schrödinger equation with the potential $q(x)$. In short, the path we take to show $\Lambda$ determines $\gamma(x)$ can be summarized in the following diagram:

$$\Lambda \longrightarrow \widetilde{h_1}(-2z_R(t),t) \longrightarrow h_0''(x,t) \longrightarrow q(x) \longrightarrow \gamma(x)$$

In Section 4, we once again use work of Nachman [18] to derive a so called $\overline{\partial}$-equation; however, our proofs of some key points are different and new (i.e. the proof of the absence of the exceptional points of $L$). In Section 5, we show that $\widetilde{h_1}''(-2z_R(t),t)$, which we obtained after the change of variables from $\widetilde{h_1}(-2z_R(t),t)$, is in fact equal to $\widetilde{h_1}(-2z_R(t),t)$. This allows us to use $\widetilde{h_1}''(-2z_R(t),t)$ and show that it determines $h_0''(x,t)$ uniquely, in Section 6, using the $\overline{\partial}$-equation. Finally, in Section 7, we show that $h_0''(x,t)$ determines $q(x)$, which in turn determines $\gamma(x)$ using the definition of $q(x)$, via $a(s(x))$ and the change of variables obtained earlier in Section 3.

## 2 $\Lambda$ determines $\widetilde{h_1}(-2z_R(t),t)$

Consider a differential operator $L(x,-i\partial_x)$ defined as follows,

$$L(x,-i\partial_x) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(\gamma^{ij}(x)\frac{\partial}{\partial x_j}) \quad \text{where } \gamma^{ij}(x) \in C^\infty(\mathbb{R}^2) \tag{2.1}$$

so we see that $Lu = 0$ is equivalent to the first equation in (1.1). Assume that

$$\sum_{i,j=1}^{2} \gamma^{ij}\xi_i\xi_j \geq \tau \sum_{i=1}^{2} \xi_i^2,\ \tau > 0,\ \text{and } \gamma^{ij}(x) = \delta^{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \text{ for } |x| > R. \tag{2.2}$$

Note that we can rewrite $L$ as follows,

$$L(x,-i\partial_x) = -\triangle + M(x,-i\partial_x) \tag{2.3}$$

where $\triangle$ is the Laplacian and $M(x,-i\partial_x) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}((\gamma^{ij}-\delta^{ij})\frac{\partial}{\partial x_j})$.
Hence it follows from (2.2) that $M(x,-i\partial_x) = 0$ when $|x| > R$.

We will need to consider the following integral equations. First, let $E(x,z)$ be Fadeev's fundamental solution of $(-i\partial_x + z(t))^2$, i.e.

$$E(x,z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{ix\cdot\eta}d\eta}{\eta\cdot\eta + 2z\cdot\eta},$$

where integral is given in distributional sense, and where $z = (t, it) \in \mathbb{C}^2$.
Assume that $h_1$ is a (compactly supported in $x$) solution of

$$h_1 + M(Eh_1) = -M(1) \tag{2.4}$$

where $M = M(x,-i\partial_x + z(t))$, and define $h_0$ by

$$h_0 = 1 + Eh_1 \tag{2.5}$$

where

$$Ef = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\tilde{f}(\eta)e^{ix\cdot\eta}d\eta}{\eta\cdot\eta + 2z\cdot\eta}$$

Using (2.4), (2.5), we obtain

$$h_0 + EMh_0 = 1. \tag{2.6}$$

**Definition** Let $L_{2,s}$ be a space with the norm $||f||^2_{L_{2,s}} = \int_{\mathbb{R}^2}|f(x)|^2(1+|x|)^{2s}dx$ (note: it is equivalent to the weighted $L^2$ space $L^2_s$ with its norm given by $||f||^2_{L^2_s} = \int_{\mathbb{R}^2}|f(x)|^2(1+|x|^2)^s dx$).

**Lemma 2.1** *i)* $(I + EM)$ *is Fredholm in* $L_{2,-\frac{1}{2}-\varepsilon}(\mathbb{R}^2)$.
*ii)* $(I + EM)h_0 = 1 \Leftrightarrow L(x,-i\partial_x + z(t))h_0 = 0$ *with* $h_0 \to 1$ *as* $|x| \to \infty$.

**Proof** :
i) To show that $I + EM$ is Fredholm, let $A(x,\xi)$ be the symbol of $I + EM$, then $A(x,\xi) = \frac{\sum_{i,j=1}^{2}\gamma^{ij}(x)\xi_i\xi_j}{|\xi|^2} + O(\frac{1}{|\xi|})$ and from the original assumption on $\gamma$, we get $|A(x,\xi)| > c$, when $|x|^2 + |\xi|^2 > N$ and hence $A(x,\xi)$ is an elliptic symbol of order 0. Hence $I + EM$ is a Fredholm in $L_{2,s}$, $\forall s$ [9].

ii) $\Leftarrow:$ We have the following sequence of equalities

$$\begin{aligned} & (M+L_0)h_0 = Lh_0 = 0 && \text{(where } L_0 = (-i\partial_x + z(t))^2) \\ \Rightarrow\ & L_0(h_0-1) = -Mh_0 && \text{(as } L_0(1)=0 \text{ using } z\cdot z = 0) \\ \Rightarrow\ & h_0 - 1 = -EMh_0 && \text{(since } E \text{ is a fundamental solution of } L_0 \text{ and} \\ & && E : L_{2,\frac{1}{2}+\varepsilon} \to L_{2,-\frac{1}{2}-\varepsilon} \text{bounded, by } (2.20)[7]) \\ \Rightarrow\ & (I+EM)h_0 = 1 \end{aligned}$$

$\Rightarrow:$ applying $L_0$ to both sides of $(I+EM)h_0 = 1$ and going in reverse of the above, we obtain $Lh_0 = 0$. To see that $h_0 \to 1$ as $|x| \to \infty$, we use estimate similar to that in [8] to get $EMh_0 = O\left(|x|^{-\frac{1}{2}}\right)$ so that using (2.6), we see that $h_0 - 1 = O\left(|x|^{-\frac{1}{2}}\right)$ hence $h_0 \to 1$ as $|x| \to \infty$, in fact $h_0 - 1 \in L_{2,-\frac{1}{2}-\varepsilon}$.

We now will follow Nachman [18] (and some ideas that go back to Novikov [19]), to show that $\Lambda$ determines $\widetilde{h_1}(-2z_R(t), t)$. Note that here the difference between our case and Nachman's is that he considers the equation with $q(x)$ a potential with compact support, while in our case we have a differential operator $M(x, -i\partial_x)$ with compact support.

**Lemma 2.2** *Given $L = -\triangle + M(x, -i\partial_x)$, $\Lambda_M$ determines $\widetilde{h_1}(-2z_R(t), t)$, where $\Lambda_M$ is a Dirichlet-to-Neumann operator, as in (1.2), and $\widetilde{h_1}(\xi, t)$ is a Fourier transform of $h_1(x, t)$, where $h_1(x, t)$ as in (2.4).*

**Proof** : We wish to show

$$(u_0\big|_{\partial\Omega}, (\Lambda_M - \Lambda_0)u\big|_{\partial\Omega}) = \int_\Omega u_0 M u dx \tag{2.7}$$

where $u$, $u_0 \in H^1(\Omega)$, $Lu = 0$, $\triangle u_0 = 0$ in $\Omega$, and $\Lambda_0$ is a Dirichlet-to-Neumann operator for $\triangle$, i.e. if

$$\begin{aligned} \triangle u_0 &= 0 \text{ in } \Omega \\ u_0\big|_{\partial\Omega} &= g, \end{aligned}$$

then

$$\Lambda_0 g = \frac{\partial u_0}{\partial n}\Big|_{\partial\Omega}.$$

Indeed, we have

$$\begin{aligned} 0 &= \int_\Omega (Lu)u_0 dx = \int_\Omega (-\triangle + M)uu_0 dx = \int_\Omega -\triangle uu_0 dx + \int_\Omega Muu_0 dx \\ &= \int_\Omega \nabla u \nabla u_0 dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} u_0 ds + \int_\Omega (Mu)u_0 dx, \end{aligned} \tag{2.8}$$

where the last equality in (2.8) is obtained by integrating the first integral by parts.
We also have using integration by parts

$$0=\int_\Omega \triangle u_0 u dx = -\int_\Omega \nabla u_0 \nabla u dx + \int_{\partial\Omega} \frac{\partial u_0}{\partial n} u ds. \tag{2.9}$$

Hence (2.8), (2.9) give

$$0=\int_\Omega (Lu)u_0 dx + \int_\Omega \triangle u_0 u dx = \int_{\partial\Omega} (\frac{\partial u_0}{\partial n}u - \frac{\partial u}{\partial n}u_0)ds + \int_\Omega (Mu)u_0 dx \tag{2.10}$$

Since $\triangle$ is a symmetric operator, note that

$$\int_{\partial\Omega} \frac{\partial u_0}{\partial n} u ds = \left(\Lambda_0 u_0\big|_{\partial\Omega}, u\big|_{\partial\Omega}\right) = \left(u_0\big|_{\partial\Omega}, \Lambda_0 u\big|_{\partial\Omega}\right). \tag{2.11}$$

Then (2.10), (2.11) give

$$\begin{aligned} 0 &= \left(u_0\big|_{\partial\Omega}, \Lambda_0 u\big|_{\partial\Omega}\right) - \left(u_0\big|_{\partial\Omega}, \Lambda_M u\big|_{\partial\Omega}\right) + \int_\Omega (Mu)u_0 dx \\ &= \left(u_0\big|_{\partial\Omega}, (\Lambda_0 - \Lambda_M)u\big|_{\partial\Omega}\right) + \int_\Omega (Mu)u_0 dx \end{aligned} \tag{2.12}$$

and (2.12) yields (2.7).
Next, we will need a few intermediate lemmas to prove Lemma 2.2; we will first state them and outline how they will be used to prove Lemma 2.2, and later we will prove them as well.

**Lemma 2.3** *Given $h_0(x,t)\big|_{\partial\Omega}$, $\widetilde{h_1}(-2z_R(t),t)$ will be given by*

$$\widetilde{h_1}(-2z_R(t),t) = -\int_{\partial\Omega} e^{ix\cdot\overline{z}}(\Lambda_M - \Lambda_0)\psi(x,t)ds \tag{2.13}$$

*where $\psi(x,t)=h_0(x,t)e^{ix\cdot z}$, $h_0$ is a solution of $L(x,-i\partial_x+z(t))h_0=0$, where $h_0\to 1$ as $|x|\to\infty$, $x=(x_1,x_2)$ and $z(t)=(t,it)$, $t\in\mathbb{C}$.*

**Lemma 2.4** *The trace on $\partial\Omega$ of the function $\psi(x,t)$ satisfies the integral equation*

$$\psi(x,t)\big|_{\partial\Omega} = e^{ix\cdot z} - S_t(\Lambda_M - \Lambda_0)(\psi(x,t)) \tag{2.14}$$

*where $S_t$ is a single layer operator corresponding to $G_t$, where*

$$G_t(x) = e^{ix\cdot z}E(x,z),$$

*$E(x,z(t))$ as before,*
*i.e. $S_t f(x) = G_t * f\big|_{\partial\Omega} = \int_{\partial\Omega} G_t(x-y)f(y)ds.$*

**Lemma 2.5** *$I+S_t(\Lambda_M-\Lambda_0)$ is a Fredholm operator: $H^s(\partial\Omega)\to H^s(\partial\Omega)$, for any s.*

**Definition:** $t$ is an exceptional point of $L$ if there exists $u \neq 0$, $u \to 0$ as $|x| \to \infty$, such that $L(x, -i\partial_x + z(t))u(x,t) = 0$ in $\mathbb{R}^2$.

**Lemma 2.6** *$\exists h$, such that $h\big|_{\partial\Omega} \neq 0$ and $(I + S_t(\Lambda_M - \Lambda_0))h\big|_{\partial\Omega} = 0$ iff $t$ is an exceptional point of $L$.*

We describe how these four lemmas reduce the proof of Lemma 2.2 to showing that $L$ has no exceptional points. Once we have shown that (2.14) holds, to solve (2.14) for $\psi(x,t)$ we would need to know that $(I + S_t(\Lambda_M - \Lambda_0))^{-1}$ exists. To do that we will use lemmas 2.5, 2.6 in the following way, first, using Lemma 2.5, we find that to show that $(I + S_t(\Lambda_M - \Lambda_0))^{-1}$ exists if and only if $t$ is not an exceptional point, it suffices by Fredholm alternative, to show that $(I + S_t(\Lambda_M - \Lambda_0))h = 0$ has a nontrivial solution $h$ iff $t$ is an exceptional point. Finally, if $L$ has no exceptional points, it will follow from Lemma 2.6 that $(I + S_t(\Lambda_M - \Lambda_0))^{-1}$ exists and, hence we can recover $\psi(x,t)$ from $\Lambda_M$. Therefore, using (2.13), we obtain the result.
So it remains to prove lemmas 2.3, 2.4, 2.5, and 2.6.

$\mathbf{Proof\ of\ Lemma\ 2.3}$ : Use (2.7) with $u = \psi(x,t)$, $u_0 = e^{ix\cdot\overline{z(t)}}$, to get

$$\left(e^{ix\cdot\overline{z(t)}}\big|_{\partial\Omega}, (\Lambda_M - \Lambda_0)\psi(x,t)\big|_{\partial\Omega}\right) = \int_\Omega e^{ix\cdot\overline{z(t)}} M(x, -i\partial_x)\psi(x,t)dx \qquad (2.15)$$

(note $\triangle e^{ix\cdot\overline{z}} = 0$, $L(x, -i\partial_x)\psi(x,t) = 0$, and $e^{ix\cdot\overline{z}}, \psi(x,t) \in H^1(\Omega)$)
Also

$$\begin{aligned} & \int_\Omega e^{ix\cdot\overline{z(t)}} M(x, -i\partial_x)\psi(x,t)dx = \\ = & \int_\Omega e^{ix\cdot\overline{z(t)}} M(x, -i\partial_x)(h_0(x,t)e^{ix\cdot z(t)})dx \\ = & \int_\Omega e^{ix\cdot\overline{z(t)}} e^{ix\cdot z(t)} M(x, -i\partial_x + z(t))(h_0(x,t))dx \\ = & \int_\Omega e^{ix\cdot 2z_R(t)} M(x, -i\partial_x + z(t))(h_0(x,t))dx \\ = & -\widetilde{h_1}(-2z_R(t), t), \end{aligned} \qquad (2.16)$$

where the last equality in (2.16) is due to $h_1 = -Mh_0$ (see (2.4), (2.5)). So (2.15) and (2.16) yield (2.13).

$\mathbf{Proof\ of\ Lemma\ 2.4}$ : Take $x \notin \overline{\Omega}$, use (2.7) with $u_0(y) = G_t(x-y)$ and $u(y) = \psi(y,t)$. Note for $y \in \overline{\Omega}$, $u_0(y)$ is smooth, $\triangle_y G_t(x-y) = 0$, $u(y) = \psi(y,t)$

satisfies $(-\triangle + M)u(y) = 0$, and $\psi(y,t) \in H^1(\Omega)$. Hence we get

$$\begin{aligned}
& [S_t(\Lambda_M - \Lambda_0)\psi(\cdot,t)](x) \\
\overset{\text{def}}{=} \quad & G_t * ((\Lambda_M - \Lambda_0)\psi(\cdot,t))\big|_{\partial\Omega} \\
= \quad & \int_{\partial\Omega} G_t(x-y)(\Lambda_M - \Lambda_0)\psi(y,t)ds \\
\overset{(2.7)}{=} \quad & \int_{\Omega} G_t(x-y)M(y,-i\partial_y)(\psi(y,t))dy
\end{aligned}$$

$$\begin{aligned}
= \quad & \int_{\Omega} G_t(x-y)e^{iy\cdot z}M(y,-i\partial_y + z(t))(h_0(y,t))dy \\
= \quad & e^{ix\cdot z}\int_{\Omega} G_t(x-y)e^{i(y-x)\cdot z(t)}M(y,-i\partial_y + z(t))(h_0(y,t))dy \\
= \quad & e^{ix\cdot z}\int_{\Omega} E(x-y,t)M(y,-i\partial_y + z(t))(h_0(y,t))dy \\
= \quad & e^{ix\cdot z}E(M(x,-i\partial_x + z(t))h_0(x,t)) \\
\overset{(2.6)}{=} \quad & e^{ix\cdot z}(1 - h_0(x,t)) = e^{ix\cdot z(t)} - \psi(x,t)
\end{aligned} \tag{2.17}$$

So (2.17) holds $\forall x \notin \overline{\Omega}$, hence restricting (2.17) to $\partial\Omega$ from the outside gives (2.14) as needed.

**Proof of Lemma 2.5** : We will first show that $S_t(\Lambda_M - \Lambda_0)$ is compact. Indeed note that by definition of $\Omega$, i.e. since $\{x : |x| \leq R\} \subset \Omega$, we have $M = 0$ near $\partial\Omega$, so the principal symbol of $\Lambda_M$ and of $\Lambda_0$ are the same, so the operator $\Lambda_M - \Lambda_0$ is of order 0 and it is bounded, also since $S_t$ a single layer operator is compact (see for example, theorems 3.2, 3.4 of [6]), it follows that $S_t(\Lambda_M - \Lambda_0)$ is compact as a product of a compact and bounded operators, as needed to be shown. Hence $I + S_t(\Lambda_M - \Lambda_0)$ is a Fredholm operator.

**Proof of Lemma 2.6** :
$\Leftarrow$: Suppose that $t$ is an exceptional point of $L$, i.e. there exists $u \neq 0$, such that

$$((-i\partial_x + z(t))^2 + M(x,-i\partial_x + z(t)))u = 0 \text{ in } \mathbb{R}^2 \tag{2.18}$$

and $u \to 0$ as $|x| \to \infty$, then $((-i\partial_x + z(t))^2)u = (-M(x,-i\partial_x + z(t)))u$ in $\mathbb{R}^2$. M has compact support, so $M(x,-i\partial_x + z(t))u \in L_{2,s}\ \forall s$, in particular $M(x,-i\partial_x + z(t))u \in L_{2,\frac{1}{2}+\varepsilon}$, and since
$E : L_{2,\frac{1}{2}+\varepsilon} \to L_{2,-\frac{1}{2}-\varepsilon}$ is bounded, we get

$$u = -E(M(x,-i\partial_x + z(t))u(x,t)).$$

Next we will show that if we let $h(x,t) = e^{ix\cdot z(t)}u(x,t)$, then this $h$ will satisfy

$$(I + S_t(\Lambda_M - \Lambda_0))\, h|_{\partial\Omega} = 0.$$

Note that (2.18) implies that $h \neq 0$ satisfies

$$L(x, -i\partial_x)h = (-\triangle + M(x, -i\partial_x))h = 0 \text{ in } \mathbb{R}^2. \tag{2.19}$$

Now, going backwards through the sequence of equalities in (2.17), we have

$$\begin{aligned} h(x,t) &= e^{ix\cdot z(t)}u(x,t) \\ &= -e^{ix\cdot z(t)}E(M(x,-i\partial_x + z(t))u(x,t)) \\ &= -e^{ix\cdot z(t)}\int_{\mathbb{R}^2} E(x-y,t)M(y,-i\partial_y + z(t))(u(y,t))dy \\ &= -e^{ix\cdot z(t)}\int_{\mathbb{R}^2} E(x-y,t)M(y,-i\partial_y)(h(y,t))e^{-iy\cdot z(t)}dy \\ &= -\int_{\mathbb{R}^2} E(x-y,t)M(y,-i\partial_y)(h(y,t))e^{i(x-y)\cdot z(t)}dy \\ &= -\int_{\mathbb{R}^2} G_t(x-y)M(y,-i\partial_y)(h(y,t))dy \\ &= -G_t * Mh(x). \end{aligned} \tag{2.20}$$

We now use (2.7) again with $u_0(y) = G_t(x-y)$ and $u(y) = h(y)$, so $\forall x \notin \overline{\Omega}$, we have

$$\begin{aligned} (S_t(\Lambda_M - \Lambda_0)h)(x) &= \int_\Omega G_t(x-y)M(y,-i\partial_y)(h(y,t))dy \\ &= G_t * Mh(x) = -h(x,t), \end{aligned} \tag{2.21}$$

so, as before, approaching $\partial\Omega$ from the outside (as in (2.17)⇒(2.14)), (2.21) yields

$$(I + S_t(\Lambda_M - \Lambda_0))h\big|_{\partial\Omega} = 0 \tag{2.22}$$

Note that $h \not\equiv 0$ on $\partial\Omega$, since if $h\big|_{\partial\Omega} = 0$, then we would have

$$\begin{aligned} (-\triangle + M)h &= 0 \text{ in } \Omega \\ h\big|_{\partial\Omega} &= 0 \end{aligned} \tag{2.23}$$

and since we are assuming that solutions of this BVP are unique, see (1.1), it follows that $h = 0$ in $\overline{\Omega}$. Then since M is supported inside $\overline{\Omega}$, (2.19) would imply that h is harmonic outside $\overline{\Omega}$. So (2.23), particularly $h\big|_{\partial\Omega} = 0$, by continuity from inside gives $\frac{\partial h}{\partial n}\big|_{\partial\Omega} = 0$, and hence the Cauchy data for $h$ is zero, so $h \equiv 0$, contradicting $h$ being a nontrivial solution of (2.19). So $h \not\equiv 0$ on $\partial\Omega$, and along with (2.22) we have shown one direction of Lemma 2.6.

⇒: Suppose $\exists h$, $h\big|_{\partial\Omega} \not\equiv 0$ satisfies (2.22). We want to show that $t$ is then an exceptional point of $L$.
Take this $h$, and let

$$v(x) = -(S_t(\Lambda_M - \Lambda_0)h)(x), \; x \in \mathbb{R}^2 \tag{2.24}$$

Let $v_-$ (respectively $v_+$) be the restrictions of $v$ to $\partial\Omega$ from inside (respectively from outside).
Then from (2.22), we get

$$v_- = h = v_+ \text{ on } \partial\Omega \tag{2.25}$$

and by jump relations for the single layer potentials (see [26], [8]),

$$\frac{\partial v_+}{\partial n} - \frac{\partial v_-}{\partial n} = (\Lambda_M - \Lambda_0)h \tag{2.26}$$

Also note that $v$ is harmonic inside $\Omega$, indeed

$$\begin{aligned}\triangle_x v &= \triangle_x(-(S_t(\Lambda_M - \Lambda_0)h)(x)) = -\triangle_x \int_{\partial\Omega} G_t(x-y)(\Lambda_M - \Lambda_0)h(y)ds \\ &= -\int_{\partial\Omega} \triangle_x G_t(x-y)(\Lambda_M - \Lambda_0)h(y)ds = 0 \end{aligned} \tag{2.27}$$

(as $x \in \Omega$, $y \in \partial\Omega$, so $x - y \neq 0$).
Now using (2.27) and (2.25), we have

$$\begin{aligned}\triangle v &= 0 \text{ in } \Omega \\ v\big|_{\partial\Omega} &= h\end{aligned} \tag{2.28}$$

So by definition of $\Lambda_0$, (2.28) gives

$$\Lambda_0 h = \frac{\partial v_-}{\partial n} \tag{2.29}$$

and substituting (2.29) into (2.26), we get

$$\frac{\partial v_+}{\partial n} = \Lambda_M h. \tag{2.30}$$

Now let

$$g = \begin{cases} P_M h & x \in \Omega \\ v & x \notin \Omega \end{cases} \tag{2.31}$$

where $P_M$ is such that $g = P_M h$ in $\Omega$ means $g$ solves

$$\begin{aligned}(-\triangle + M)g &= 0 \text{ in } \Omega \\ g\big|_{\partial\Omega} &= h\end{aligned}$$

(also note that therefore $\frac{\partial g}{\partial n}\big|_{\partial\Omega} = \Lambda_M h$ by definition of $\Lambda_M$)
Hence on $\partial\Omega$, we get

$$g_- = h = v_+ = g_+ \tag{2.32}$$

and

$$\frac{\partial g_-}{\partial n} = \Lambda_M h = \frac{\partial v_+}{\partial n} = \frac{\partial g_+}{\partial n} \tag{2.33}$$

where first and last equalities in (2.32), (2.33) are obtained from (2.31), and the middle equality in (2.32), (2.33) are by (2.25) and (2.30).
So (2.31), (2.32), (2.33) and $\triangle v = 0$ outside $\Omega$ imply that $g$ solves

$$(-\triangle + M)g = 0 \text{ in } \mathbb{R}^2. \tag{2.34}$$

Recall, to show that $t$ is an exceptional point of $L$, need to show $\exists$ a function $u \neq 0$ solving (2.18) and such that $u \to 0$ as $|x| \to \infty$. Let $u = ge^{-ix\cdot z(t)}$, then (2.34) shows that $u$ in fact solves (2.18) in $\mathbb{R}^2$, so remains to show that $u \to 0$. Note from definition of $g$ (2.31), $g = v$ when $x \notin \Omega$. From (2.24), we get

$$\begin{aligned} u(x,t) = v(x)e^{-ix\cdot z(t)} &= -e^{-ix\cdot z(t)}(S_t(\Lambda_M - \Lambda_0)h)(x) \\ &= -E(M(x, -i\partial_x + z(t))(P_M h e^{-ix\cdot z(t)})), \end{aligned}$$

where the last equality follows similarly to (2.17). Since once again $M$ has compact support $E(M(x, -i\partial_x + z(t))(P_M h e^{-ix\cdot z(t)})) = O(\frac{1}{|x|^{\frac{1}{2}}})$,
so $u = O(\frac{1}{|x|^{\frac{1}{2}}})$, hence $u \to 0$ as $|x| \to \infty$, (and $u \in L_{2,-\frac{1}{2}-\varepsilon}$).

## 3 Change of variables

**Lemma 3.1** *Given $Lu = 0$, where $L$ as in (2.1), there is a change of coordinates $y = s(x) = (s_1(x), s_2(x))$, such that*

$$L(x, -i\partial_x)u(x) = L'(y, -i\partial_y)u'(y)|_{y=s(x)} = L''(y, -i\partial_y)u''(y)|_{y=s(x)} \tag{3.1}$$

*where*

$$\begin{aligned} u'(s(x)) &= u(x) \\ u''(y) &= a^{1/2}(y)u'(y), \end{aligned}$$

$$L'(y, -i\partial_y) = -\frac{1}{\left|\frac{dx}{dy}\right|}\sum_{i=1}^{2}\frac{\partial}{\partial y_i}(a(y)\frac{\partial}{\partial y_i}) \tag{3.2}$$

$$L''(y, -i\partial_y) = \frac{a^{1/2}(y)}{\left|\frac{dx}{dy}\right|}\left(-\triangle + \frac{\triangle a^{1/2}(y)}{a^{1/2}(y)}\right) \tag{3.3}$$

*where*

$$a(y) = \sum_{i,j=1}^{2}\gamma^{ij}(s^{-1}(y))\frac{\partial s_1}{\partial x_i}\frac{\partial s_1}{\partial x_j}\left|\frac{dx}{dy}\right| \stackrel{\text{def}}{=} \langle\nabla s_1, \nabla s_1\rangle\left|\frac{dx}{dy}\right|$$

*and*

$$\left|\frac{dx}{dy}\right| = J_{s^{-1}(y)} = \frac{1}{\begin{vmatrix} \frac{\partial s_1}{\partial x_1} & \frac{\partial s_1}{\partial x_2} \\ \frac{\partial s_2}{\partial x_1} & \frac{\partial s_2}{\partial x_2} \end{vmatrix}}.$$

**Proof** : We will show that there is a change of coordinates $y = s(x)$, such that $\forall u, v \in C_0^\infty(\mathbb{R}^2)$

$$\big(L(x,-i\partial_x)u(x)\circ s^{-1}(y), v(y)\big) = (L'(y,-i\partial_y)u'(y), v(y))$$

First consider,

$$(L(x,-i\partial_x)u(x), v(x)) = \int_{\mathbb{R}^2}\left(-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}(\gamma^{ij}\frac{\partial}{\partial x_j})u(x)\right)v(x)dx$$

integrating by parts we get

$$\int_{\mathbb{R}^2}\sum_{i,j=1}^{2}\gamma^{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx$$

So

$$\begin{aligned}
&\big(L(x,-i\partial_x)u(x)\circ s^{-1}(y), v(y)\big) = \\
= \quad & \sum_{i,j=1}^{2}\int_{\mathbb{R}^2}\gamma^{ij}(s^{-1}(y))\sum_{k=1}^{2}\frac{\partial u'}{\partial y_k}\frac{\partial s_k}{\partial x_j}\sum_{p=1}^{2}\frac{\partial v}{\partial y_p}\frac{\partial s_p}{\partial x_i}\left|\frac{dx}{dy}\right|dy \\
= \quad & \int_{\mathbb{R}^2}\sum_{k,p=1}^{2}\Big(\sum_{i,j=1}^{2}\gamma^{ij}(s^{-1}(y))\frac{\partial s_k}{\partial x_j}\frac{\partial s_p}{\partial x_i}\Big)\frac{\partial u'}{\partial y_k}\frac{\partial v}{\partial y_p}\left|\frac{dx}{dy}\right|dy \qquad (3.4)
\end{aligned}$$

We will use the notation $\langle f, g\rangle = \sum_{i,j=1}^{2}\gamma^{ij}f_i g_j$, so if we require that $y = s(x)$ make the $y$-coordinates isothermal, we need to have

$$\begin{aligned}
\langle\nabla s_1, \nabla s_1\rangle &= \langle\nabla s_2, \nabla s_2\rangle \\
\langle\nabla s_1, \nabla s_2\rangle &= 0 \qquad (3.5)
\end{aligned}$$

We will show later that, in fact, such change of coordinates exists, but assuming it for now, and continuing the computation of $\big(L(x,-i\partial_x)u(x)\circ s^{-1}(y), v(y)\big)$, we find by (3.4), (3.5) that

$$\begin{aligned}
&\big(L(x,-i\partial_x)u(x)\circ s^{-1}(y), v(y)\big) \\
= \quad & \int_{\mathbb{R}^2}\langle\nabla s_1, \nabla s_1\rangle\sum_{i=1}^{2}\frac{\partial u'}{\partial y_i}\frac{\partial v}{\partial y_i}\left|\frac{dx}{dy}\right|dy \\
= \quad & \sum_{i=1}^{2}\int_{\mathbb{R}^2}\left(\langle\nabla s_1, \nabla s_1\rangle\left|\frac{dx}{dy}\right|\right)\frac{\partial u'}{\partial y_i}\frac{\partial v}{\partial y_i}dy \qquad (3.6)
\end{aligned}$$

and integrating the first integral of the (3.6) by parts, we get

$$\begin{aligned}
&\big(L(x,-i\partial_x)u(x)\circ s^{-1}(y), v(y)\big) \qquad (3.7) \\
= \quad & \int_{\mathbb{R}^2}\left(-\sum_{i=1}^{2}\frac{\partial}{\partial y_i}\left(\langle\nabla s_1, \nabla s_1\rangle\left|\frac{dx}{dy}\right|\right)\frac{\partial u'}{\partial y_i}\right)v(y)dy
\end{aligned}$$

Now (3.4), (3.7) give $L(x, -i\partial_x)u(x) = L'(y, -i\partial_y)u'(y)$, where $L'(y, -i\partial_y)$ is the operator in (3.2).

We now will show that there is a change of coordinate $y = s(x)$, satisfying (3.5). We will employ the method in [7]. We are solving the system (3.5). Letting $\psi = s_1 + is_2$, (3.5) could be written as

$$\langle \nabla\psi, \nabla\psi \rangle = 0 \tag{3.8}$$

factoring (3.8), we get

$$\gamma^{11}\left(\frac{\partial\psi}{\partial x_1} - \lambda\frac{\partial\psi}{\partial x_2}\right)\left(\frac{\partial\psi}{\partial x_1} - \overline{\lambda}\frac{\partial\psi}{\partial x_2}\right) = 0$$

where

$$\lambda = \frac{-i\sqrt{\gamma^{11}\gamma^{22} - (\gamma^{12})^2} - \gamma^{12}}{\gamma^{11}}$$

Note that $\lambda = -i$, when $|x| > R$.
To solve (3.8), we will look for solutions $g(x)$ of $\psi(x) = x_1 + ix_2 + g(x)$, where $g(x) \to 0$ as $|x| \to \infty$, and simply solve

$$\frac{\partial\psi}{\partial x_1} - \lambda\frac{\partial\psi}{\partial x_2} = 0$$

by finding $g$ such that

$$\frac{\partial g}{\partial x_1} - \lambda\frac{\partial g}{\partial x_2} = i\lambda - 1. \tag{3.9}$$

Note, that we can rewrite (3.9) as

$$\frac{\partial g}{\partial \overline{z}} - \mu\frac{\partial g}{\partial z} = \mu \tag{3.10}$$

where

$$\mu = \frac{\sqrt{\gamma^{11}\gamma^{22} - (\gamma^{12})^2} - \gamma^{11} - i\gamma^{12}}{\sqrt{\gamma^{11}\gamma^{22} - (\gamma^{12})^2} + \gamma^{11} - i\gamma^{12}}$$

Note that $|\mu| < 1$ and $\mu = 0$ when $|x| > R$.
**Remark**: Though, we will not use this until Section 7, it is worth noticing that, since $\lambda = -i$, when $|x| > R$

$$\frac{\partial\psi}{\partial x_1} - \lambda\frac{\partial\psi}{\partial x_2} = 0$$

reduces to

$$2\frac{\partial\psi}{\partial \overline{z}} = \frac{\partial\psi}{\partial x_1} + i\frac{\partial\psi}{\partial x_2} = 0, \text{ when } |x| > R$$

and so $\psi(z)$ is therefore analytic when $|z| > R$.
Now using Ahlfors' method in [1], we find a solution $g$ to (3.10), such that

$$D^\alpha g = O\left(\frac{1}{|z|^{\alpha+1}}\right) \text{ as } |z| \to \infty,\ |\alpha| \leq 1. \tag{3.11}$$

It also follows from [1] that $y = s(x) = (Re\ \psi, Im\ \psi)$ is a one-to-one map with nonvanishing Jacobian. Hence there is a change of coordinates $y = s(x)$ satisfying (3.5).

We now will derive the second equality in (3.1).
Let $u''(y) = a^{1/2}(y)u'(y)$, then

$$\frac{\partial}{\partial y_i}\left(a\frac{\partial u'}{\partial y_i}\right) = \left(\frac{a^{-3/2}}{4}\left(\frac{\partial a}{\partial y_i}\right)^2 - \frac{a^{-1/2}}{2}\frac{\partial^2 a}{\partial y_i^2}\right)u''(y) + a^{1/2}\frac{\partial^2 u''(y)}{\partial y_i^2} \tag{3.12}$$

So using $\triangle a^{1/2} = \sum_{i=1}^{2} -\frac{a^{-3/2}}{4}\left(\frac{\partial a}{\partial y_i}\right)^2 + \frac{a^{-1/2}}{2}\frac{\partial^2 a}{\partial y_i^2}$ and (3.12), we get

$$\sum_{i=1}^{2}\frac{\partial}{\partial y_i}\left(a\frac{\partial u'}{\partial y_i}\right) = -\triangle a^{1/2}u''(y) + a^{1/2}\triangle u''(y) \tag{3.13}$$

so

$$L'(y, -i\partial_y)u'(y) = \frac{a^{1/2}(y)}{\left|\frac{dx}{dy}\right|}\left(-\triangle u'' + \frac{\triangle a^{1/2}(y)}{a^{1/2}(y)}u''\right)$$

This shows the second equality in (3.1)
Note, if we call

$$q(y) = \frac{\triangle a^{1/2}(y)}{a^{1/2}(y)} \tag{3.14}$$

then

$$0 = L''u'' = \frac{a^{1/2}(y)}{\left|\frac{dx}{dy}\right|}\left(-\triangle + q(y)\right)u''(y),$$

hence

$$(-\triangle + q(y))u''(y) = 0 \tag{3.15}$$

Let's look closer at $a(y)$. Recall, $a(y) = \sum_{i,j=1}^{2}\gamma^{ij}(s^{-1}(y))\frac{\partial s_1}{\partial x_i}\frac{\partial s_1}{\partial x_j}\left|\frac{dx}{dy}\right|$.
From (3.11), we know that

$$g(z) = O\left(\frac{1}{|z|}\right),\ \frac{\partial g}{\partial z} = O\left(\frac{1}{|z|^2}\right) \text{ as } |z| \to \infty.$$

and $y = s(x) = (Re\ \psi, Im\ \psi),$ where $\psi(z) = z + g(z),$
so $\psi(z) = z + O\left(\frac{1}{|z|}\right)$ and $\frac{\partial\psi}{\partial z} = 1 + O\left(\frac{1}{|z|^2}\right)$.
Since we showed earlier that $\psi(z)$ is analytic when $|z| > R$, we know that Cauchy-Riemann equations are satisfied, i.e

$$\frac{\partial s_1}{\partial x_1} = \frac{\partial s_2}{\partial x_2} \quad \text{and} \quad \frac{\partial s_1}{\partial x_2} = -\frac{\partial s_2}{\partial x_1}. \tag{3.16}$$

Now, when $|x| > R,\ |y| > R$ (as by above $s(x) = x + O\left(\frac{1}{|x|}\right)$) and hence $\gamma^{ij}(s^{-1}(y)) = \delta^{ij}$, then using (3.16), we get

$$a(y) = \left(\left(\frac{\partial s_1}{\partial x_1}\right)^2 + \left(\frac{\partial s_1}{\partial x_2}\right)^2\right)\left(\frac{\partial s_1}{\partial x_1}\frac{\partial s_2}{\partial x_2} - \frac{\partial s_1}{\partial x_2}\frac{\partial s_2}{\partial x_1}\right)^{-1} \overset{(3.16)}{=} 1 \quad \text{when } |y| > R. \tag{3.17}$$

# 4 The $\overline{\partial}$-equation

Our goal in this section is to show that $h_0(x,t)$ satisfies the following $\overline{\partial}$-equation

$$\frac{\partial h_0}{\partial \bar{t}} = -\frac{e^{-2ix\cdot z_R(t)}}{4\pi\bar{t}}\tilde{h}_1(-2z_R(t),t)\overline{h_0(x,t)} \tag{4.1}$$

To do this we will need the following two lemmas.

**Lemma 4.1** *$(I + EM)^{-1}$ exists if and only if $(I + EM')^{-1}$ exists, where $M = M(x, -i\partial_x + z(t))$ and $M(x, -i\partial_x)$ as in (2.3), $M' = M'(y, -i\partial_y + z(t))$ and $M'(y, -i\partial_y) = \triangle + L'$, where $L'$ as in (3.2).*

**Proof** : Using Lemma 2.1, and similar results for $L'$ and $M'$, it is enough to show that $\exists!\ h_0$ a solution of $L(x, -i\partial_x + z(t))h_0 = 0$, such that $h_0 \to 1$ as $|x| \to \infty$, if and only if $\exists!\ h_0'$ a solution of $L'(y, -i\partial_y + z(t))h_0' = 0$, such that $h_0' \to 1$ as $|y| \to \infty$.
We start from

$$L(x, -i\partial_x + z(t))h_0 = 0, \quad \text{with } h_0 \to 1 \text{ as } |x| \to \infty \tag{4.2}$$

multiplying (4.2) by $e^{ix\cdot z(t)}$ and after a simple computation, we get

$$L(x, -i\partial_x)(e^{ix\cdot z(t)}h_0(x,t)) = e^{ix\cdot z(t)}L(x, -i\partial_x + z(t))(h_0(x,t)) = 0 \tag{4.3}$$

Now changing variables $y = s(x)$ in (4.3) gives

$$L'(y, -i\partial_y)(e^{is^{-1}(y)\cdot z(t)}h_0(s^{-1}(y),t)) = 0 \tag{4.4}$$

Again multiplying (4.4) by $e^{-iy\cdot z(t)}$, we get

$$L'(y, -i\partial_y + z(t))(e^{i(s^{-1}(y)-y)\cdot z(t)}h_0(s^{-1}(y),t)) = 0 \tag{4.5}$$

and set

$$h_0'(y,t) = e^{i(s^{-1}(y)-y)\cdot z(t)} h_0(s^{-1}(y),t) \tag{4.6}$$

Then since $s^{-1}(y) = y + O(\frac{1}{|y|})$ as $|y| \to \infty$, $h_0'(y,t) \to h_0(s^{-1}(y,t))$ as $|y| \to \infty$. Hence as $|y| \to \infty$, we find that $h_0'(y,t) \to 1$. So $h_0'(y,t)$ is a solution of $L'(y, -i\partial_y + z(t))h_0' = 0$, satisfying $h_0' \to 1$ as $|y| \to \infty$.
We can similarly show, starting with a solution $u$ of $L'(y, -i\partial_y + z(t))u = 0$, $u \to 1$ as $|y| \to \infty$, that we can obtain a solution $v$ of $L(x, -i\partial_x + z(t))v = 0$, $v \to 1$ as $|x| \to \infty$.
Also note that $h_0$ is unique if and only if $h_0'$ is; this is immediate from the relation (4.6).

**Lemma 4.2** *$I + EM'$ is invertible for $t \neq 0$.*

**Proof** : By Lemma 2.1 we only need to show that

$$L'(y, -i\partial_y + z(t))g = 0, \quad g \to 0 \text{ as } |y| \to \infty \tag{4.7}$$

implies that $g \equiv 0$.
From (3.2) we get

$$\frac{\left|\frac{dx}{dy}\right|}{a(y)} L'(y, -i\partial_y) = -\frac{\sum_{i=1}^{2} \frac{\partial}{\partial y_i}(a(y)\frac{\partial}{\partial y_i})}{a(y)} = -\triangle - \frac{\sum_{i=1}^{2} \frac{\partial a}{\partial y_i}\frac{\partial}{\partial y_i}}{a(y)} \tag{4.8}$$

So to show (4.7) implies $g = 0$, it is enough by (4.8) to show that

$$\left( (-i\partial_y + z(t))^2 - \frac{\sum_{j=1}^{2} \frac{\partial a}{\partial y_j}\left(\frac{\partial}{\partial y_j} + iz_j(t)\right)}{a(y)} \right) g = 0, \text{ with } g \to 0 \text{ as } |y| \to \infty \tag{4.9}$$

implies that $g \equiv 0$.
We can rewrite the equation (4.9), as

$$\left( (-i\partial_y + z(t))^2 - i\sum_{j=1}^{2} A_j \left(-i\frac{\partial}{\partial y_j} + z_j(t)\right) \right) g = 0$$

where $A_j = \frac{\frac{\partial a}{\partial y_j}}{a(y)}$. Note that $A_j(y) \in \mathbb{R}$.
Let $g_1 = e^{iy\cdot z_R(t)}g$, then $g_1$ satisfies (by computation similar to (4.2)⇒(4.3))

$$\left( (-i\partial_y + i\, Im\; z(t))^2 - i\sum_{j=1}^{2} A_j \left(-i\frac{\partial}{\partial y_j} + i\, Im\; z_j(t)\right) \right) g_1 = 0 \tag{4.10}$$

Recall $z(t) = (t, it) = (t_1, -t_2) + i(t_2, t_1)$, so $Im\; z(t) = (t_2, t_1)$, and (4.10) becomes

$$\left( \sum_{j=1}^{2} \left(-i\frac{\partial}{\partial y_j} + it_{j'}\right)^2 - i\sum_{j=1}^{2} A_j \left(-i\frac{\partial}{\partial y_j} + it_{j'}\right) \right) g_1 = 0 \tag{4.11}$$

where

$$j' = \begin{cases} 1, & \text{when } j = 2 \\ 2, & \text{when } j = 1 \end{cases}$$

Simplifying (4.11), we get

$$-\left(\frac{\partial}{\partial y_1} - t_2\right)^2 g_1 - \left(\frac{\partial}{\partial y_2} - t_1\right)^2 g_1 - A_1\left(\frac{\partial}{\partial y_1} - t_2\right) g_1 - A_2\left(\frac{\partial}{\partial y_2} - t_1\right) g_1 = 0. \tag{4.12}$$

Next, we let $v_1 = \left(\frac{\partial}{\partial y_1} - t_2\right) g_1,\ v_2 = \left(\frac{\partial}{\partial y_2} - t_1\right) g_1 = 0.$
Note that

$$\left(\frac{\partial}{\partial y_2} - t_1\right) v_1 = \left(\frac{\partial}{\partial y_1} - t_2\right) v_2 \tag{4.13}$$

Substituting $v_1$ and $v_2$, (4.12) becomes

$$-\left(\frac{\partial}{\partial y_1} - t_2\right) v_1 - \left(\frac{\partial}{\partial y_2} - t_1\right) v_2 - A_1 v_1 - A_2 v_2 = 0 \tag{4.14}$$

Now letting $w = v_1 + iv_2$, and using (4.13), (4.14) gives

$$\left(\frac{\partial}{\partial y_1} - t_2\right) w - i\left(\frac{\partial}{\partial y_2} - t_1\right) w + A_1\frac{w + \overline{w}}{2} + A_2\frac{w - \overline{w}}{2i} = 0$$

so

$$\left(\frac{\partial}{\partial y_1} + it_1\right) w - i\left(\frac{\partial}{\partial y_2} - it_2\right) w + A_1' w + A_2' \overline{w} = 0 \tag{4.15}$$

where

$$A_1' = \frac{A_1 - iA_2}{2}, \quad A_2' = \frac{A_1 + iA_2}{2}$$

Now letting $w_1 = e^{i(y_1t_1 - y_2t_2)}w$, (4.15) gives

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}\right) w_1 + A_1' w_1 + A_2' \overline{w}_1 \left(e^{2i(y_1t_1 - y_2t_2)}\right) \\ &= \left(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}\right) w_1 + \left(A_1' + e^{2i(y_1t_1 - y_2t_2)} A_2' \frac{\overline{w}_1}{w_1}\right) w_1 \end{aligned} \tag{4.16}$$

Note that as $g \to 0$, $g_1 = e^{iy\cdot z_R(t)}g \to 0$, when $|y| \to \infty$, also note that $\frac{\partial g}{\partial y_i} \to 0$. Indeed, $g + EMg = 0$, so

$$\frac{\partial g}{\partial y_i} = -\frac{\partial}{\partial y_i}(EMg) = -E\frac{\partial}{\partial y_i}(Mg) = -EM\frac{\partial g}{\partial y_i} - EM_i g \tag{4.17}$$

where second equality in (4.17) holds because $E(Mg)$ is a convolution of $E$ and $Mg$, and $M_i$ stands for operator obtained from $M$ by differentiating its coefficients in $y_i$. Now since $M$ has compactly supported coefficients and $E$ is a convolution with a function which goes to zero as $|y| \to \infty$, it follows from

(4.17), that $\frac{\partial g}{\partial y_i} \to 0$ as $|y| \to \infty$. Hence $\frac{\partial g_1}{\partial y_i} \to 0$.
Therefore,

$$\begin{aligned} w &= v_1 + iv_2 = \left(\frac{\partial}{\partial y_1} - t_2\right) g_1 + i\left(\frac{\partial}{\partial y_2} - t_1\right) g_1 \\ &= \left(\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}\right) g_1 - i(t_1 - it_2)g_1 \to 0 \text{ as } |y| \to \infty. \end{aligned}$$

Hence $w_1 = e^{i(y_1t_1 - y_2t_2)}w \to 0$ as $|y| \to \infty$ as well.
We now will prove the following

**Claim 4.3** *Suppose* $\frac{\partial w}{\partial \bar{z}} + B(x)w = 0$*, where* $B$ *is bounded and compactly supported. If* $w \to 0$ *as* $|z| \to \infty$*, then* $w \equiv 0$*.*

This is due to Bers-Vekua [25], sometimes called a Vanishing Lemma.

**Proof** : Take $\psi$, such that $\frac{\partial \psi}{\partial \bar{z}} = B$, i.e.

$$\psi(z) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{B(y_1, y_2)}{z - \xi} dy_1\, dy_2,$$

where $\xi = y_1 + iy_2, \quad z = x_1 + ix_2$.
Note that $\psi \to 0$ as $|z| \to \infty$ because $B$ has compact support.
Now let $f = we^{\psi}$, then

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial w}{\partial \bar{z}}e^{\psi} + we^{\psi}\frac{\partial \psi}{\partial \bar{z}} = 0, \quad \forall z$$

so $f$ is entire.
And as $\psi \to 0$, and $w \to 0$ by assumption, we get $f \to 0$ as $|z| \to \infty$.
So by Liouville's Theorem, $f \equiv 0$. Hence $w \equiv 0$, so Claim 4.3 is proved.

Now, recall we had equation (4.16), which could be written as

$$\frac{\partial}{\partial z} w_1 + Bw_1 = 0 \tag{4.18}$$

where we used $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}\right)$ and

$$B = \frac{1}{2}\left(A_1'(y) + A_2'(y)e^{2i(y_1t_1 - y_2t_2)}\frac{\overline{w_1}}{w_1}\right) \tag{4.19}$$

Note that taking complex conjugate of (4.18), we get

$$\frac{\partial}{\partial \bar{z}}\overline{w}_1 + \overline{B}\overline{w}_1 = 0$$

So the same conclusion holds for equation (4.18) as for the one in Claim 4.3. Hence to conclude that $w_1 \equiv 0$ in (4.18), all we have left to show is that $B$ is bounded and with compact support.
Indeed, from (4.19) we get

$$|B| \le C\left(|A_1'| + |A_2'|\right) \le \tilde{C}\left(|A_1| + |A_2|\right), \quad C, \tilde{C} > 0 \text{ constants.}$$

Recall $A_j = \dfrac{\frac{\partial a}{\partial y_j}}{a(y)}$, $a(y)$ is smooth and by (3.17) $a(y) = 1$ when $|y| > R$. Hence $A_i(y) = 0$ when $|y| > R$, so $B$ is continuous with compact support, so bounded, and therefore, by Claim 4.3 it follows that $w_1 \equiv 0$.
So $w = e^{i(y_1 t_1 - y_2 t_2)} w_1 = 0$. Hence $v_1 = v_2 = 0$, and therefore

$$\left(\frac{\partial}{\partial y_1} - t_2\right) g_1 = 0 = \left(\frac{\partial}{\partial y_2} - t_1\right) g_1$$

Taking Fourier transform in $y_i$, we get

$$(i\xi_1 - t_2)\tilde{g}_1(\xi_1, \xi_2) = 0 = (i\xi_2 - t_1)\tilde{g}_1(\xi_1, \xi_2) \tag{4.20}$$

If $t \neq 0$, then either $t_1 \neq 0$ or $t_2 \neq 0$, so (4.20) yields $\tilde{g}_1(\xi_1, \xi_2) = 0$, and hence $g_1(y) = 0$, so $g(y) = e^{-iy \cdot z_R(t)} g_1(y) = 0$.

We will now state and prove the main result of this section.

**Lemma 4.4** *The function $h_0(x, t)$ satisfies the $\overline{\partial}$-equation (4.1) for $t \neq 0$.*

**Proof** : We start by taking $\dfrac{\partial}{\partial \bar{t}}$ of (2.6), to get

$$\frac{\partial h_0}{\partial \bar{t}} + \frac{\partial}{\partial \bar{t}}(EMh_0) = 0 \tag{4.21}$$

Using Lemma 2.1 of [7], we know that if

$$I(t) = \int_{\mathbb{R}^2} \frac{f(\eta) d\eta_1 d\eta_2}{\eta \cdot \eta + 2z(t) \cdot \eta}$$

then

$$\frac{\partial}{\partial \bar{t}}(I(t)) = \frac{1}{2}\left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)(I(t)) = -\frac{\pi}{\bar{t}} f(-2z_R(t)) \tag{4.22}$$

We will now compute $\dfrac{\partial}{\partial \bar{t}}(EMh_0)$:

$$\begin{aligned} \frac{\partial}{\partial \bar{t}}(EMh_0) &= \frac{\partial}{\partial \bar{t}}\left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\widetilde{Mh_0} e^{ix\cdot\eta} d\eta_1 d\eta_2}{\eta \cdot \eta + 2z(t) \cdot \eta}\right) \\ &= \int_{\mathbb{R}^2} \frac{\widetilde{M \frac{\partial h_0}{\partial \bar{t}}} e^{ix\cdot\eta} d\eta_1 d\eta_2}{\eta \cdot \eta + 2z(t) \cdot \eta} - \frac{\pi}{4\pi^2 \bar{t}} \widetilde{Mh_0}(-2z_R(t), t) e^{-2ix\cdot z_R(t)}, \end{aligned} \tag{4.23}$$

where to obtain the last equality in (4.23) we used $\frac{\partial}{\partial \bar{t}}M = 0$ as $M$ is a polynomial in $z(t)$ and $z(t) = (t, it)$ is analytic in t, and equation (4.22) with $f(\eta) = \widetilde{Mh_0}(\eta, t)e^{ix\cdot\eta}$.
So (4.23), using the definition of $Ef$, becomes

$$\frac{\partial}{\partial \bar{t}}(EMh_0) = EM\frac{\partial h_0}{\partial \bar{t}} + \frac{e^{-2ix\cdot z_R(t)}}{4\pi\bar{t}}\widetilde{h_1}(-2z_R(t), t) \tag{4.24}$$

where we used

$$Mh_0 = -h_1 \tag{4.25}$$

which follows from (2.4), (2.5).
Hence substituting (4.24) into (4.21), we get

$$\frac{\partial h_0}{\partial \bar{t}} + EM\frac{\partial h_0}{\partial \bar{t}} = -\frac{e^{-2ix\cdot z_R(t)}}{4\pi\bar{t}}\widetilde{h_1}(-2z_R(t), t). \tag{4.26}$$

Now using lemmas 4.1, 4.2, it follows that $(I + EM)^{-1}$ exists for $t \neq 0$, so (4.26) gives

$$\begin{aligned}\frac{\partial h_0}{\partial \bar{t}} &= (I + EM)^{-1}\left(-\frac{e^{-2ix\cdot z_R(t)}}{4\pi\bar{t}}\widetilde{h_1}(-2z_R(t), t)\right) \\ &= -\frac{\widetilde{h_1}(-2z_R(t), t)}{4\pi\bar{t}}(I + EM)^{-1}(e^{-2ix\cdot z_R(t)}),\end{aligned} \tag{4.27}$$

where the last equality in (4.27) holds because $(I + EM)^{-1}$ is an operator in $x$ only.
Now to compute $(I + EM)^{-1}(e^{-2ix\cdot z_R(t)})$ we will proceed as in [7].
Using (2.6) and (4.25), we obtain

$$h_0(x, t) = 1 + \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{\widetilde{h_1}(\eta, t)e^{ix\cdot\eta}d\eta}{\eta\cdot\eta + 2z(t)\cdot\eta}. \tag{4.28}$$

Taking complex conjugate of (4.28), we get

$$\overline{h_0(x, t)} = 1 + \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{\overline{\widetilde{h_1}(\eta, t)}e^{-ix\cdot\eta}d\eta}{\eta\cdot\eta + 2\overline{z(t)}\cdot\eta}, \tag{4.29}$$

then changing variables $-\eta' = \eta + 2z_R(t)$ in (4.29) gives

$$\overline{h_0(x, t)} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{\overline{\widetilde{h_1}(-\eta' - 2z_R(t), t)}e^{-ix\cdot(-\eta' - 2z_R(t))}d\eta'}{(\eta' + z(t))^2} + 1, \tag{4.30}$$

and multiplying (4.30) by $e^{-2ix\cdot z_R(t)}$, we obtain

$$\overline{h_0(x, t)}e^{-2ix\cdot z_R(t)} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{\overline{\widetilde{h_1}(-\eta' - 2z_R(t), t)}e^{ix\cdot\eta'}d\eta'}{(\eta' + z(t))^2} + e^{-2ix\cdot z_R(t)} \tag{4.31}$$

We next prove the following:

**Claim 4.5** $\overline{\widetilde{h_1}(-\eta' - 2z_R(t), t)} = \mathcal{F}\left(-M(x, -i\partial_x + z(t))(\overline{h_0(x,t)}e^{-2ix\cdot z_R(t)})\right)$, *where $\mathcal{F}(u)$ is the Fourier transform of $u$.*

**Proof of Claim** : From (2.4), (2.5), $h_1(x,t) = -M(x, -i\partial_x + z(t))h_0(x,t)$, so $\widetilde{h_1}(\eta, t) = -\int_{\mathbb{R}^2} M(x, -i\partial_x + z(t))(h_0(x,t))e^{-ix\cdot\eta}dx$, hence

$$\overline{\widetilde{h_1}(\eta, t)} = -\int_{\mathbb{R}^2} M(x, i\partial_x + \overline{z(t)})(\overline{h_0(x,t)})e^{ix\cdot\eta}dx, \tag{4.32}$$

changing variables $-\eta' = \eta + 2z_R(t)$ in (4.32) gives

$$\begin{aligned} \overline{\widetilde{h_1}(-\eta' - 2z_R(t), t)} &= -\int_{\mathbb{R}^2} M(x, i\partial_x + \overline{z(t)})(\overline{h_0(x,t)})e^{-ix\cdot(\eta'+2z_R(t))}dx \\ &= \mathcal{F}\left(-e^{-2ix\cdot z_R(t)}M(x, i\partial_x + \overline{z(t)})(\overline{h_0(x,t)})\right) \end{aligned} \tag{4.33}$$

Note that we have the following sequence of equalities:

$$\begin{aligned} & e^{-2ix\cdot z_R(t)}M(x, i\partial_x + \overline{z(t)})(\overline{h_0(x,t)}) \\ = \ & M(x, i\partial_x - z(t))(\overline{h_0(x,t)}e^{-2ix\cdot z_R(t)}) \\ = \ & M(x, -i\partial_x + z(t))(\overline{h_0(x,t)}e^{-2ix\cdot z_R(t)}), \end{aligned} \tag{4.34}$$

where the last equality in (4.34) follows from the definition of $M$, i.e. because $M(x,\xi) = M(x, -\xi)$.
So (4.33) and (4.34) yield Claim 4.5.

Now using (4.31) and Claim 4.5, we get

$$\begin{aligned} & e^{-2ix\cdot z_R(t)} - \overline{h_0(x,t)}e^{-2ix\cdot z_R(t)} \\ = \ & \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\mathcal{F}\left(M(x, -i\partial_x + z(t))(\overline{h_0(x,t)}e^{-2ix\cdot z_R(t)})\right)e^{ix\cdot\eta'}d\eta'}{(\eta' + z(t))^2} \end{aligned}$$

which is simply, by the definition of $Ef$,

$$\overline{h_0(x,t)}e^{-2ix\cdot z_R(t)} + E\left(M\left(\overline{h_0(x,t)}e^{-2ix\cdot z_R(t)}\right)\right) = e^{-2ix\cdot z_R(t)}, \tag{4.35}$$

and since $(I + EM)^{-1}$ exists for $t \neq 0$, (4.35) gives

$$\overline{h_0(x,t)}e^{-2ix\cdot z_R(t)} = (I + EM)^{-1}\left(e^{-2ix\cdot z_R(t)}\right). \tag{4.36}$$

Hence (4.36) and (4.27) yield (4.1) as needed.

# 5 $\widetilde{h_1}(-2z_R(t), t) = \widetilde{h_1'}(-2z_R(t), t) = \widetilde{h_1''}(-2z_R(t), t)$

Similarly to Lemma 4.4, we can show that the same $\overline{\partial}$-equation holds for $h_0'$ and $h_0''$ as for $h_0$, where

$$\begin{aligned} L(x, -i\partial_x + z(t))h_0 &= 0, \quad h_0 \to 1 \text{ as } |x| \to \infty \\ L'(y, -i\partial_y + z(t))h_0' &= 0, \quad h_0' \to 1 \text{ as } |y| \to \infty \\ L''(y, -i\partial_y + z(t))h_0'' &= 0, \quad h_0'' \to 1 \text{ as } |y| \to \infty \end{aligned}$$

where $L,\ L',\ L''$ as in (3.1), (3.2), (3.3), i.e. we can show that

$$\frac{\partial h_0'(y,t)}{\partial \bar{t}} = -\frac{e^{-2iy\cdot z_R(t)}}{4\pi\bar{t}} \widetilde{h_1'}(-2z_R(t),t)\overline{h_0'(y,t)} \tag{5.1}$$

and

$$\frac{\partial h_0''(y,t)}{\partial \bar{t}} = -\frac{e^{-2iy\cdot z_R(t)}}{4\pi\bar{t}} \widetilde{h_1''}(-2z_R(t),t)\overline{h_0''(y,t)} \tag{5.2}$$

hold, where $h_1'$, $h_0'$ are defined by (2.4), (2.5) with $M$ replaced by $M'$. Likewise, $h_1''$, $h_0''$ are defined by (2.4), (2.5) with $M$ replaced by $M''$, where $M''(y,-i\partial_y) = \frac{\left|\frac{dx}{dy}\right|}{a^{1/2}(y)} L''(y,-i\partial_y) + \triangle$, i.e. $M''(y,-i\partial_y) = q(y)$ with $q(y)$ as in (3.14).

**Lemma 5.1** $\widetilde{h_1}(-2z_R(t),t) = \widetilde{h_1'}(-2z_R(t),t)$.

**Proof** : We have

$$\frac{\partial}{\partial \bar{t}}\left(h_0(x,t)e^{ix\cdot z(t)}\right) = e^{ix\cdot z(t)}\left(-\frac{e^{-2ix\cdot z_R(t)}}{4\pi\bar{t}}\widetilde{h_1}(-2z_R(t),t)\overline{h_0(x,t)}\right)$$
$$= \quad -\frac{e^{-ix\cdot \bar{z}(t)}}{4\pi\bar{t}}\widetilde{h_1}(-2z_R(t),t)\overline{h_0(x,t)} = -\frac{\widetilde{h_1}(-2z_R(t),t)}{4\pi\bar{t}}\overline{h_0(x,t)e^{ix\cdot z(t)}} \tag{5.3}$$

where to obtain the first equality in (5.3), we used (4.1) and that $z(t)$ is analytic in $t$ gives $\frac{\partial}{\partial \bar{t}}(e^{ix\cdot z(t)}) = 0$.
Using (5.1), (5.2) we can get a similar relation to (5.3) for $h_0'(y,t)$ and for $h_0''(y,t)$.
Now using relation (4.6) between $h_0$ and $h_0'$, we get

$$h_0'(y,t)e^{iy\cdot z(t)} = h_0(s^{-1}(y),t)e^{is^{-1}(y)\cdot z(t)} \tag{5.4}$$

Hence

$$\frac{\partial}{\partial \bar{t}}\left(h_0'(y,t)e^{iy\cdot z(t)}\right) = \frac{\partial}{\partial \bar{t}}\left(h_0(s^{-1}(y),t)e^{is^{-1}(y)\cdot z(t)}\right) \tag{5.5}$$

Now using (5.3) for $h_0(x)$, where $x = s^{-1}(y)$, we get

$$\frac{\partial}{\partial \bar{t}}\left(h_0(s^{-1}(y),t)e^{is^{-1}(y)\cdot z(t)}\right) =$$
$$= \quad -\frac{\widetilde{h_1}(-2z_R(t),t)}{4\pi\bar{t}}\overline{h_0(s^{-1}(y),t)e^{is^{-1}(y)\cdot z(t)}} \tag{5.6}$$

Then using a similar relation to (5.3) for $h_0'(y)$, we get

$$\frac{\partial}{\partial \bar{t}}\left(h_0'(y,t)e^{iy\cdot z(t)}\right) = -\frac{\widetilde{h_1'}(-2z_R(t),t)}{4\pi\bar{t}}\overline{h_0'(y,t)e^{iy\cdot z(t)}} \tag{5.7}$$

Hence (5.4), (5.5), (5.6), (5.7) give

$$\widetilde{h_1}(-2z_R(t),t) = \widetilde{h_1'}(-2z_R(t),t),$$

which completes the proof of Lemma 5.1.

We also want to show the following relation between $h_0''(y)$ and $h_0(x)$:

$$h_0''(y,t) = e^{i(s^{-1}(y)-y)\cdot z(t)} h_0(s^{-1}(y),t) a^{1/2}(y). \tag{5.8}$$

Indeed we have

$$\begin{aligned} & e^{ix\cdot z(t)} L(x, -i\partial_x + z(t))(h_0(x,t)) = \\ = \quad & L(x, -i\partial_x)(e^{ix\cdot z(t)} h_0(x,t)) \end{aligned}$$

Recall from(4.4), that if $L(x,-i\partial_x+z(t))h_0 = 0$, then $L'(y,-i\partial_y)(e^{is^{-1}(y)\cdot z(t)} h_0(s^{-1}(y),t)) = 0$, this was after the first change of coordinates, but after second change of variables $u''(y) = a^{1/2}(y)u'(y)$, we would simply have

$$L''(y,-i\partial_y)(e^{is^{-1}(y)\cdot z(t)} h_0(s^{-1}(y),t) a^{1/2}(y)) = 0 \tag{5.9}$$

and as before (5.9) will imply that

$$L''(y,-i\partial_y + z(t))(e^{i(s^{-1}(y)-y)\cdot z(t)} h_0(s^{-1}(y),t) a^{1/2}(y)) = 0 \tag{5.10}$$

So we get the relation (5.8), and since $s^{-1}(y) = y + O\left(\frac{1}{|y|}\right)$, so as $|y| \to \infty$, $e^{i(s^{-1}(y)-y)\cdot z(t)} h_0(s^{-1}(y),t) \to h_0(s^{-1}(y),t)$, and also since $a(y) = 1$ when $|y| > R$ (by (3.17)), we see from (5.8), that $h_0''(y,t) \to 1$ also as $|y| \to \infty$.
So then we can use the relation between $h_0$ and $h_0''$ in (5.8) to show that $\widetilde{h_1}(-2z_R(t),t) = \widetilde{h_1''}(-2z_R(t),t)$ just as we did in Lemma 5.1 for $\widetilde{h_1}$ and $\widetilde{h_1'}$.

Hence the main result of this section holds:

$$\widetilde{h_1}(-2z_R(t),t) = \widetilde{h_1'}(-2z_R(t),t) = \widetilde{h_1''}(-2z_R(t),t) \tag{5.11}$$

# 6 Uniqueness of the solution of $L_p''(y, -i\partial + z(t))u = 0$

Suppose we have $L_p$, $p = 1,2$ as in (1.3) and the corresponding Dirichlet-to-Neumann maps, $\Lambda_p$, defined in (1.2). Then Lemma 2.5 says that if $\Lambda_1 = \Lambda_2$, then $\widetilde{h_1^{(1)}}(-2z_R(t),t) = \widetilde{h_1^{(2)}}(-2z_R(t),t)$. Hence using (5.11), we get

$$\widetilde{h_1'^{(1)}}(-2z_R(t),t) = \widetilde{h_1'^{(2)}}(-2z_R(t),t) \tag{6.1}$$

and

$$\widetilde{h_1''^{(1)}}(-2z_R(t),t) = \widetilde{h_1''^{(2)}}(-2z_R(t),t) \tag{6.2}$$

where $h_1'^{(p)}$ corresponds to $L_p'(y,-i\partial_y + z(t))$, $p = 1,2$
and $h_1''^{(p)}$ corresponds to $L_p''(y,-i\partial_y + z(t))$, $p = 1,2$.

**Lemma 6.1** *If* $L_p''(y, -i\partial_y + z(t))h_0''^{(p)}(y,t) = 0$ *with* $h_0''^{(p)} \to 1$ *as* $|y| \to \infty$, $p = 1, 2$. *Then* $\widetilde{h_1''^{(1)}}(-2z_R(t), t) = \widetilde{h_1''^{(2)}}(-2z_R(t), t)$ *implies that* $h_0''^{(1)}(y,t) = h_0''^{(2)}(y,t)$.

**Proof** : Let $h(y,t) = h_0''^{(1)}(y,t) - h_0''^{(2)}(y,t)$. Since $h_0''^{(p)}$ satisfies the same $\overline{\partial}$-equation (4.1), we obtain

$$\frac{\partial h}{\partial \bar{t}} = -\frac{e^{-2iy\cdot z_R(t)}}{4\pi\bar{t}}\widetilde{h_1}(-2z_R(t), t)\overline{h(y,t)} \tag{6.3}$$

as

$$\widetilde{h_1}(-2z_R(t), t) = \widetilde{h_1''^{(p)}}(-2z_R(t), t),$$

$p = 1, 2,$ by (5.11) and (6.2).

Let

$$B(y,t) = -\frac{e^{-2iy\cdot z_R(t)}}{4\pi\bar{t}}\widetilde{h_1}(-2z_R(t), t)\frac{\overline{h(y,t)}}{h(y,t)}, \tag{6.4}$$

so (6.3) becomes

$$\frac{\partial h}{\partial \bar{t}} = Bh \tag{6.5}$$

We wish to show that $h \equiv 0$. To do that, consider

$$\psi(y,t) = \Pi B = -\frac{1}{\pi}\int \frac{B(y,t')}{t - t'}dt', \tag{6.6}$$

i.e. where $\Pi$ is a fundamental solution operator of $\dfrac{\partial}{\partial \bar{t}}$, so

$$\frac{\partial \psi}{\partial \bar{t}} = B$$

Let $f = he^{-\psi}$, then $\dfrac{\partial f}{\partial \bar{t}} = 0$.

Note, from (6.4), $B$ has a singularity at $t = 0$, and since $\dfrac{\partial f}{\partial \bar{t}} = 0$, we see that $f$ is analytic, when $t \neq 0$.

**Lemma 6.2** $t = 0$ *is a removable singularity of* $f$.

Let's assume this lemma for now, then $f$ would be analytic $\forall t$, so $f$ is entire. So if we could show that $f \to 0$ as $|t| \to \infty$, then we would invoke Liouville's Theorem, to conclude that $f \equiv 0$. Hence we will first prove

**Lemma 6.3** $f \to 0$ *as* $|t| \to \infty$.

**Proof of Lemma 6.3** : By [7], (2.20), $h(y,t) = h_0''^{(1)}(y,t) - h_0''^{(2)}(y,t) \to 0$ as $t \to \infty$. Since $f = he^{-\psi}$, to show that $f \to 0$ enough to show that $\psi \to 0$ as $t \to \infty$.

Recall the relation (6.6) between $\psi$ and $B$. We will use a computation similar to that in [7]:

$$\int_{\mathbb{R}^2} \frac{B(y,t')}{t-t'}dt' = \int_{\mathbb{R}^2} \frac{A(y,t')}{\overline{t'}(t-t')}dt',$$

where

$$A(y,t) = -\frac{e^{-2iy\cdot z_R(t)}}{4\pi}\widetilde{h_1}(-2z_R(t),t)\frac{\overline{h(y,t)}}{h(y,t)}, \tag{6.7}$$

so

$$\left|\int_{\mathbb{R}^2} \frac{B(y,t')}{t-t'}dt'\right| \le C\left(\int_{\mathbb{R}^2} \frac{dt'}{(|t'||t-t'|)^p}\right)^{1/p}\left(\int_{\mathbb{R}^2} |A(y,t')|^r dt'\right)^{1/r}$$

where $r = \frac{2}{1-\delta}$, $p = \frac{r}{r-1}$, $0 < \delta < 1$, by Holdër inequality.
After a change of variables $t' = |t|w$, we get

$$\int_{\mathbb{R}^2} \frac{dt'}{(|t'||t-t'|)^p} = \frac{|t|^2}{|t|^{2p}}\int_{\mathbb{R}^2} \frac{dw}{(|w||w-\frac{t}{|t|}|)^p} \le \frac{C}{|t|^{2(p-1)}}$$

where the inequality above is due to $1 < p = \frac{2}{1+\delta} < 2$, for $0 < \delta < 1$.
So to show $\psi(y,t) \to 0$ as $|t| \to \infty$, enough to show that $A(y,t) \in L_r$ as a function of $t$.
Now $|A(y,t)| = C|\widetilde{h_1}(-2z_R(t),t)|$, from (6.7) and

$$\widetilde{h_1}(\xi,t) = -\int_{\mathbb{R}^2} q(y)h_0''(y,t)e^{-iy\cdot\xi}dy, \tag{6.8}$$

since $M''(y,-i\partial_y) = L''(y,-i\partial_y) + \triangle$ and by (3.3) $M''(y,-i\partial_y)$ is simply a multiplication by $q(y)$, where from (3.14)

$$q(y) = \frac{\triangle a^{1/2}(y)}{a^{1/2}(y)}.$$

Recall from (3.17), that $a(y) = 1$, when $|y| > R$, so $q(y) = 0$ when $|y| > R$, and $q(y)$ is smooth.
We can rewrite (6.8) as

$$\widetilde{h_1}(\xi,t) = -\int_{\mathbb{R}^2} q(y)e^{-iy\cdot\xi}dy + \int_{\mathbb{R}^2} q(y)(1-h_0''(y,t))e^{-iy\cdot\xi}dy, \tag{6.9}$$

so we have

$$\widetilde{h_1}(\xi,t) = -\tilde{q}(\xi) + \int_{\mathbb{R}^2} q(y)(1-h_0''(y,t))e^{-iy\cdot\xi}dy.$$

By [7], (2.20),

$$|h_0''(y,t) - 1| \le \frac{C}{|t|^{2/r_1}},\ r_1 > 2,\ |t| > M,$$

so

$$\left| \int_{\mathbb{R}^2} q(y)(1 - h_0''(y,t)) e^{-iy\cdot\xi} dy \right| \le \frac{C}{|t|^{2/r_1}}.$$

Note the last inequality is due to $q(y)$ being continuous with compact support and hence $\int_{\mathbb{R}^2} |q(y)|dy$ is therefore just a constant.
Also since $q(y)$ is smooth and with compact support, it follows that $\tilde{q}(\xi) \in L_{r_2},\ \forall r_2 > 0$. Also note that since $z_R(t) = t_1 - it_2$, so when $\xi = -2z_R(t)$, $|\xi| = 2|t|$. Therefore, $\tilde{q}(\xi) \in L_{r_2},\ r_2 > 0$ implies $\tilde{q}(-2z_R(t)) \in L_{r_2},\ r_2 > 0$ in $t$. Hence together with the above estimates by taking $r_1,\ r_2$ such that $r > r_1$ and $r = r_2$ it follows that $|\widetilde{h_1}(-2z_R(t), t)| \in L_r$.
Which shows that $\psi \to 0$ as $|t| \to \infty$. Hence $f = he^{-\psi} \to 0$ as $|t| \to \infty$, which completes the proof of Lemma 6.3.

**Proof of Lemma 6.2** :

We will use (6.3) with polar coordinates

$$t = re^{i\varphi}, \quad \eta = (\rho\cos\theta, \rho\sin\theta),$$

then, denoting $h(x,t) = h(x,r,\varphi)$, we get

$$\left( \frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi} \right) h(x,r,\varphi) = a(x,r,\varphi)\overline{h}(x,r,\varphi) \tag{6.10}$$

where

$$a(x,r,\varphi) = -\frac{1}{2\pi r} e^{-2ix\cdot r(\cos\varphi, -\sin\varphi)} \widetilde{h_1}(-2r(\cos\varphi, -\sin\varphi), r, \varphi). \tag{6.11}$$

Note, $\dfrac{a\overline{h}}{h} = B(x,r,\varphi)$, where $B(x,t)$ as in (6.4).
Note that $M''(x, -i\partial_x) = q(x)$, where $q(x)$ as in (3.14). We will show in Lemma 6.4 that $(I + M''E_0)^{-1}$ exists, and will assume it here. Using (6.11), we find that, since we may replace $\widetilde{h_1}(-2r(\cos\varphi, -\sin\varphi), r, \varphi)$ by the Fourier transform at 0,

$$a(x,r,\varphi) = -\frac{1}{2\pi r}\left( \int_{\mathbb{R}^2} h_1(x,r,\varphi)dx + O(r) \right), \quad \text{as } r \to 0.$$

To determine the behavior of $a$ as $r \to 0$ we need information on $h_1$. The fundamental solution $E(x,t)$ has the following asymptotics when $r \to 0$ ([18]):

$$E(x,t) = E_0(x) + \frac{1}{2\pi}\ln r + E_1(x,r,\varphi), \tag{6.12}$$

where

$$E_0(x) = \frac{1}{2\pi}\ln|x| + C_0$$

$C_0$ is constant, $|E_1(x,r,\varphi)| \le C|r|^{1-\varepsilon}$, $0 < \varepsilon < 1$.
Substituting this into (2.4) with $M$ replaced by $M''$, the assumption that $(I +$

$M''E_0)^{-1}$ exists leads to two cases,
case 1.

$$a(x,r,\varphi) = -\frac{c}{r(1-c\ln r)} + O\left(\frac{1}{r^{1-\varepsilon_1}}\right), \varepsilon_1 > 0,\ 0 < r < \varepsilon, \tag{6.13}$$

case 2.

$$|a(x,r,\varphi)| \leq Cr^{-\varepsilon_1}, \quad 1 > \varepsilon_1 > 0, \text{ when } 0 < r < \varepsilon. \tag{6.14}$$

**Lemma 6.4** $(I + M''E_0)^{-1}$ *exists.*

**Proof** : Note, this is equivalent to showing that $(I + M'E_0)^{-1}$ exists (see Lemma 4.1). Consider $g + M'E_0 g = 0$, to show $g = 0$. Note here $M' = M'(x, -i\partial_x)$.
Since $M'$ has compact support in $\Omega$, $g = 0$ for $x \in \mathbb{R}^2 \backslash \Omega$.
Let $v = E_0 g$, then

$$(\triangle + M')v = 0, \tag{6.15}$$

since $L' = \triangle + M'$, $L'$ as in (4.8), (6.15) gives

$$\triangle v + \sum_{i=1}^{2} A_i \frac{\partial v}{\partial x_i} = 0, \quad \text{in } \mathbb{R}^2, \tag{6.16}$$

where $A_i = \frac{\frac{\partial a}{\partial y_i}}{a(y)}$, recall $A_i = 0$, when $|x| > R$ by (3.17).
We shall show that $v = 0$. We have,

$$v = E_0 g = \int_\Omega \left(\frac{1}{2\pi}\ln|x-y| + C_0\right) g(y)dy. \tag{6.17}$$

We will consider two cases $\int_\Omega g(y)dy = 0$ and $\int_\Omega g(y)dy \neq 0$.
Note (6.17) gives

$$v = (\frac{1}{2\pi}\ln|x| + C_0)\int_\Omega g(y)dy + O\left(\frac{1}{|x|}\right), \quad \text{when } |x| \to \infty. \tag{6.18}$$

Case 1: $\int_\Omega g(y)dy = 0$, then by (6.18), $v = O\left(\frac{1}{|x|}\right)$, when $|x| \to \infty$. And hence, by Maximum principle, $v \equiv 0$.

Case 2: $\int_\Omega g(y)dy \neq 0$, then $\int_\Omega g(y)dy = c$, $c$=constant,
if $c > 0$, then $v \to +\infty$ by (6.18), and so $v$ would have a local minimum, but since it satisfies (6.16), it would imply that $v$ is constant, a contradiction.
if $c < 0$, then $v \to -\infty$, by (6.18), so $v$ would have a local maximum, and hence again we get a contradiction.

Hence $v \equiv 0$, and then $g = \triangle v = 0,\ \forall x \in \mathbb{R}^2$ as needed.

We now return to the proof of Lemma 6.2.
In case 2. (6.14), $|B| = |a| = O(r^{-\epsilon_1})$, and $\psi = \Pi B$ is bounded near $r = 0$. In case 1. (6.13), $B(x,r,\phi) = B_1(r) + B_2(x,r,\phi)$, where $B_1 = -\dfrac{c}{r(1-c\ln r)}$ and $B_2 = O(r^{-1+\epsilon_1})$. We will construct $\psi$ in this case as $\psi = \psi_1(r) + \psi_2(x,r,\phi)$, where

$$\psi_1 = \int_0^r \frac{cd\rho}{-\rho(1-c\ln\rho)} = O(\ln|\ln r|).$$

and $\psi_2 = \Pi B_2$. Then $\psi = O(\ln|\ln r|) = O(\ln|\ln|t||)$ and $\dfrac{\partial\psi}{\partial\bar{t}} = B$. Hence $e^{-\psi} = O(\ln|t|)$ as $|t| \to 0$, and $f = he^{-\psi}$ satisfies $\dfrac{\partial f}{\partial \bar{t}} = 0$. Substituting (6.12) into (2.4) with $M$ replaced by $M''$ as before, and using $h_0 = 1 + Eh_1$, one can show that $h = O(\ln r) = O(\ln|t|)$ as well. So we get $f = he^{-\psi} = O(\ln^2|t|) = O\left(\dfrac{1}{|t|^{1-\varepsilon}}\right)$, $\varepsilon > 0$, as $t \to 0$. And since $f$ is analytic in $0 < |t| < \delta$, it follows that $t = 0$ is a removable singularity of $f$, which completes the proof of Lemma 6.2.
So lemmas 6.2, 6.3, and Liouville's theorem imply that $f \equiv 0$. Hence $h = fe^{\psi} \equiv 0$, which completes the proof of Lemma 6.1.

# 7 Proof of Theorem 1

We now are ready to conclude our proof of Theorem 1.
We are back to our region $\Omega$, so we extend $\gamma(x)$ from $\Omega$ to $\mathbb{R}^2$, so that $\gamma(x)$ is smooth on all of $\mathbb{R}^2$ and $\gamma^{ij}(x) = \delta^{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ for large $x$ .
Using Lemma 6.1 and the definition of $M''(y, -i\partial_y) = q(y)$, where $L''(y, -i\partial_y)$ as in (3.3), we have

$$((-i\partial_y + z(t))^2 + q_p(y))h_0''^{(p)}(y,t) = 0 \qquad (7.1)$$

with

$$h_0''^{(1)}(y,t) = h_0''^{(2)}(y,t).$$

Then defining $u_p = e^{iz\cdot y}h_0''^{(p)}$, we get

$$(-\triangle + q_p(y))u_p = 0 \text{ with } u_1 = u_2, \qquad (7.2)$$

by multiplying (7.1) by $e^{iy\cdot z(t)}$.
Hence, (7.2) gives

$$q_1 = \frac{\triangle u_1}{u_1} = \frac{\triangle u_2}{u_2} = q_2$$

Now from the definition of $q(y)$,

$$q_p(y) = \frac{\triangle a_p^{1/2}(y)}{a_p^{1/2}(y)}, \quad p = 1, 2 \qquad (7.3)$$

where

$$a_p(y) = \frac{\sum_{i,j=1}^{2} \gamma_p^{ij}(s^{(p)^{-1}}(y)) \frac{\partial s_1^{(p)}}{\partial x_i} \frac{\partial s_1^{(p)}}{\partial x_j}}{\left|\frac{ds^{(p)}}{dx}\right|} = \frac{\langle \triangle s_1^{(p)}, \triangle s_1^{(p)} \rangle}{\left|\frac{ds^{(p)}}{dx}\right|}, \quad p = 1, 2.$$

Let

$$\alpha_p = \frac{\triangle a_p^{1/2}(y)}{a_p^{1/2}(y)}, \; p = 1, 2. \tag{7.4}$$

Then from the assumption of Theorem 1, we have

$$\Lambda_1 = \Lambda_2$$

And we have shown thus far that it implies that

$$q_1(y) = q_2(y).$$

So we have

$$\frac{\triangle a_1^{1/2}(y)}{a_1^{1/2}(y)} = \frac{\triangle a_2^{1/2}(y)}{a_2^{1/2}(y)} \tag{7.5}$$

We will show that (7.5) implies $a_1 = a_2$. Indeed, let

$$a = a_1^{1/2} - a_2^{1/2},$$

then $a$ satisfies

$$\triangle a + ga = 0, \text{ where } g = \frac{\triangle a_1^{1/2}(y)}{a_1^{1/2}(y)} = \frac{\triangle a_2^{1/2}(y)}{a_2^{1/2}(y)}.$$

Recall that $a_i = 1$ when $|y| > R$ by (3.17), so $a = 0$ when $|y| > R$, hence $a\big|_{\partial\Omega} = 0$ and $\frac{\partial a}{\partial n}\big|_{\partial\Omega} = 0$, and hence by Uniqueness of the Cauchy problem, $a \equiv 0$, and so $a_1 = a_2$.

Now $a_1 = a_2$, using the definition of $a_i$ gives

$$\frac{\sum_{i,j=1}^{2} \gamma_1^{ij}(s^{(1)^{-1}}(y)) \frac{\partial s_1^{(1)}}{\partial x_i} \frac{\partial s_1^{(1)}}{\partial x_j}}{\left|\frac{ds^{(1)}}{dx}\right|} = \frac{\sum_{i,j=1}^{2} \gamma_2^{ij}(s^{(2)^{-1}}(y)) \frac{\partial s_1^{(2)}}{\partial x_i} \frac{\partial s_1^{(2)}}{\partial x_j}}{\left|\frac{ds^{(2)}}{dx}\right|}$$

or

$$\frac{(J_{S^{(1)}})^T \gamma_1 (J_{S^{(1)}})}{det(J_{S^{(1)}})} = \frac{(J_{S^{(2)}})^T \gamma_2 (J_{S^{(2)}})}{det(J_{S^{(2)}})} \tag{7.6}$$

Let $S = (s^{(2)})^{-1} \circ s^{(1)}$, then (7.6) becomes

$$\gamma_2 = \frac{(J_S)^T \gamma_1 J_S}{det(J_S)} \circ S^{-1} \tag{7.7}$$

It only remains to show that

$$S = I \text{ when } |x| > R.$$

Note that in the equation (1.3), for $p = 1$ the highest degree term is $-\triangle$ when $|x| > R$, and after change of variables $S = (s^{(2)})^{-1} \circ s^{(1)}$, we still have the highest degree term (see equation (1.3), $p = 2$) $-\triangle$ when $|x| > R$. So we call $w = f(z)$, where $f(z)$ is a change of variables that takes us from equation (1.3), $p = 1$ to the equation (1.3), $p = 2$, where $w = u + iv$, and $z = x + iy$. We get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h(x, y) \tag{7.8}$$

and then after the change of variables,

$$\frac{1}{|f'(z)|^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \check{h}(u, v) \tag{7.9}$$

where $\check{h}(u, v) = h(f(x, y))$, and where (7.9) is since $\psi^i(z),\ i = 1, 2$ is analytic when $|z| > R$ by the **Remark** in Section 3, hence so is $f(z) = (\psi^{(2)})^{-1}\psi^{(1)}(z)$, so we have from (7.8), (7.9)

$$|f'(z)|^2 = 1 \ \Rightarrow\ f'(z) = e^{i\theta} \overset{\psi(z)\to z}{\Rightarrow} w = f(z) = z \tag{7.10}$$

and therefore, $S = I$ when $|x| > R$, hence $S|_{\partial\Omega} = I$, which completes our Proof.

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