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A critical reading on the theory of gravitational waves propagation

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Abstract

In this paper a discussion arises about causality in Relativity leading to argue inherent issues. After that, by presenting formal analogies between General Relativity and continuum lagrangian description, there follows a hypothetical approach for deriving the equations of gravitational radiation based on the application of the two fundamental quadratic forms of Space-Time (see equ. (3.6)). Moreover, a technique of resolution is touched. As last expectation, the paper would want to bring substantial novelties in matter of genesis and especially of gravitational waves detection.

1. Introduction

It is evident that electromagnetism and gravitation have some likeness, such as the variation with the inverse squared of Newton's gravitation law and Coulomb's law and as pointed out from several authors between the 4-density of current and the energy-momentum tensor, the 4-potential and the metric tensor, as well as between Einstein's equation and Poisson's equation to which the former is led back in Newtonian approximation [1]. We remind these ones:

Einstein's equation:

$$(1.1) \qquad G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = -\frac{8\pi G}{c^4}T_\mu^\nu \qquad (\mu,\nu=0,1,2,3)$$

Poisson's equation:

$$(1.2) \qquad |\nabla|^2 V = 4\pi\rho$$

If we pay attention we may notice that Einstein's equation does not possess the important character of *causality* as well as Poisson's equation which describes the field generated by a mass distribution. Instead there have such a character in classical physics the Newton's equation of dynamics or Faraday's law which tie an effect (the acceleration or the current) to a cause (the force or the variation of flow of magnetic field). Indeed, if we focus on electromagnetism we see that Poisson's equation is not sufficient in order to derive the equations of electromagnetic wave propagation. On the other hand, gravitational radiation is usually derived from (1.1) in linear approximation so that the classical waves equation turns out [2]:

$$(1.3a) \qquad \partial_t^2 h^{\mu\nu} - c^2(\partial_x^2+\partial_y^2+\partial_z^2)h^{\mu\nu}=0$$

with $h^{\mu\nu}$ small perturbations of Minkowski space geometry. In (1.3a) it is evident that the time variable puts on a privileged role in spite of the spatial ones. But if we deal with the time variable in its space character as usually made in relativity we cannot obtain an equation of the following type on using Einstein's equation alone (by ∂_x^2 symbol we mean the second derivative with respect to X):

$$(1.3b) \qquad \partial_\tau^2 h^{\mu\nu} - c^2(\partial_{ct}^2+\partial_x^2+\partial_y^2+\partial_z^2)h^{\mu\nu}=0,$$

with τ proper time

Moreover, by a purely mathematical point of view, waves are considered among the characteristic varieties [3,4] of discontinuity of equ.(1.1), but again our attention concentrate on the equation, not on its discontinuities; consequently I shall attempt an alternative approach. Alternatives to general Relativity have been made in order to avoid singularities in Einstein's theory or to overcome the difficulties of admitting the existence of dark matter or even to replace inflationary cosmology. Interesting among these there are the theories containing scalar, vector or tensor fields naturally besides the metric tensor [5,6,7]. The following approach is quite different inasmuch as it introduces a tensor derived from the metric tensor but possesses a single gravitational field and supposes Einstein's equation without replacing it except to describe the framework of gravitational radiation.

2. The two fundamental quadratic forms of Space-Time

In order to support our reasoning we shall refer to the lagrangian description of continuum mechanics, which introduces the concept (as well-known in differential geometry) of first and second quadratic fundamental forms associated to a riemannian space characterized by lagrangian metric [8]. We recall these elementary concepts.

An ipersurface embedded into an N-dimensional Euclidean space may be represented in vectorial form, by assigning the vector connecting a prefixed point O to its generic surface point P, in function of the parameters y^α characterizing the degrees of freedom of the ipersurface $(\alpha=1,2,\ldots,N-1)$. In the case of ordinary ipersurfaces $(N=2)$ we have $OP=OP(y^1,y^2)$. Then, let e_1,e_2 be the vectors tangent to the surface according to the lines $y^2=const$ and $y^1=const$:

$$(2.1)\qquad\qquad e_\alpha=\frac{\partial OP}{\partial y^\alpha}\qquad\qquad (\alpha=1,2)$$

which individualize a local base for each point of the surface; therefore it makes sense to consider the direct product $e_\alpha\cdot e_\beta=g_{\alpha\beta}$, coefficients of the metric of the surface σ. From the symmetry of the direct product the coefficients of the metric also are symmetric. It is well-known in differential geometry that the second quadratic form $k_{\alpha\beta}$ of a surface in the ordinary space defines the differential (increment) of metric $g_{\alpha\beta}$ in passing from this to a parallel surface [3]. In fact, let OP be a point of the surface and OP' be a point of the parallel surface defined by: $OP'=OP+\tau\nu,$ with ν normal versor to the surface and τ a prefixed parameter: we mean to say that at less of second order infinitesimals in τ we have:

$$\partial g_{\alpha\beta}=g'_{\alpha\beta}(OP')-g_{\alpha\beta}(OP)=2\tau k_{\alpha\beta}(OP)\qquad\qquad(\alpha,\beta=1,2)$$

By taking advantage of the analogy between surfaces geometry and riemannian Space-Time at four dimensions therefore let us take:

$g_{\mu\nu}$ Space-Time metric tensor and first fundamental quadratic form and

$$(2.2)\qquad\qquad k_{\mu\nu}=\frac{1}{2c}\frac{\partial g_{\mu\nu}}{\partial\tau}$$

deformation velocity and second fundamental quadratic form (c is the speed of light);

The derivative in (2.2) is with respect to the proper time and both the quadratic forms are function of the cronotope coordinates x^ρ and indirectly of the proper time, i.e. $g_{\mu\nu}(x^\rho/\tau)$ and $k_{\mu\nu}(x^\rho/\tau)$. For construction $k_{\mu\nu}$ is symmetric too. Notice that x^ρ and τ are not independent and $g_{\mu\nu}(x^\rho/\tau)$ stands for the metric of Space-Time associated to the reference frame passing through the cronotope point x^ρ just at proper time τ in its evolution. Concerning deformation velocity in gravitation then the following reasoning makes sense: let a reference frame be free falling, then I can always define locally geodetic coordinates of constant metric tensor $\eta_{\mu\nu}$s in a point of the cronotope through which the frame passes so that the Christoffel symbols of the second kind $\Gamma^{\sigma}{}_{\mu\nu}$ are null; hence, we may wonder: how will the metric tensor vary along the geodetic line after the time τ? This is denoted by $k_{\mu\nu}$ or better by its integral over τ. In fact, on examining the finite deformation undergone by the free falling frame, i.e. the local deformation of the metric from the cronotope point P_0 at 0 proper time to the point P at τ proper time, we have:

$$(2.3) \qquad 2c\int_0^{\tau} k_{\mu\nu}(x^{\rho}/\tau')d\tau' = g_{\mu\nu}(x^{\rho}/\tau) - \eta_{\mu\nu} = \chi_{\mu\nu}(x^{\rho}/\tau)$$

We call $\chi_{\mu\nu}(x^{\rho}/\tau)$ local characteristics of deformation as in continuum mechanics. To the first order with regard to the cronotope coordinates, the $\chi_{\mu\nu}(x^{\rho}/\tau)$ are the known $h_{\mu\nu}(x^{\rho}/\tau)$ of gravitation theory, which we could call linear characteristics of deformation in analogy with continuum. In (2.2) it is plain that if the metric is constant along the universe line of the particle then there is no deformation. Thus, we may associate $k_{\mu\nu}(x^{\rho}/\tau)$ to a frame in free fall such as the Earth or better a gravitational waves detector on it. Hence, as in classical mechanics it is necessary to know position and velocity in order to characterize the dynamics of a system, therefore, two parameters, *thus characterization of the evolution of metric associated to a surface and to Space-Time in particular also needs two parameters and a dynamic equation, while Einstein's equation concerns only static description (or stationary at the most) of gravitational field.*

3. Equations of consistency of the two quadratic forms of relativistic Space-Time

Then the following problem has to be attacked: given a system of gravitating masses, how associate their dynamic properties to $k_{\mu\nu}$? Again continuum mechanics might come to us in aid. Really in the lagrangian description of continuum there exist a remarkable expression of the angular velocity tensor ω_{hk} (called the vortex), function just of the deformation velocity [8] (∇_j stand for the covariant derivative with respect to the j coordinate):

$$(3.1) \qquad \nabla_j\omega_{hk} = \nabla_h k_{kj} - \nabla_k k_{hj} \qquad (h,k=1,2,3) \qquad \text{with}$$

$$k_{hk} = \frac{\partial g_{hk}}{\partial t},$$

g_{hk} lagrangian metric and $\omega_{hk}=1/2(\partial_h v_k - \partial_k v_h)$, where v_h are the components of the lagrangian velocity.
Furthermore, the vortex possesses the following property:

$$(3.2) \qquad \nabla_j\omega_{hk} + \nabla_h\omega_{kj} + \nabla_k\omega_{jh} = 0$$

We can see in (3.1) that if at any given time the tensor k_{hk} is identically null, ω_{hk} does not depend on the particle considered, i.e., $\omega_{hk} = \text{const}$ in the continuum and the act of motion is rigid. Equ. (3.1) may be generalized without too many difficulties to the Space-Time by considering it a 4-dimensional continuum, so in $\omega_{\mu\nu}=1/2(\partial_\mu v_\nu - \partial_\nu v_\mu)$ v_μ are the components of the 4-velocity of a punctual free falling object. In order to find an antisymmetric tensor in gravitation like $\omega_{\mu\nu}$ we may cleverly contract the Riemann tensor by the antisymmetric tensor $\varepsilon^{\alpha\beta}$ so defined[i]:

$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha} \text{ and } \qquad \varepsilon^{\alpha\beta}=1 \qquad \text{for } \alpha,\beta \text{ consecutive indexes } \quad (\alpha,\beta=0,1,2,3).$$

We name it contraction tensor. Its tensorial character comes out from the following decomposition of $\varepsilon^{\alpha\beta}$:
$\varepsilon^{\alpha\beta} = \delta^{\alpha\beta}{}_{12} + \delta^{\alpha\beta}{}_{23} + \delta^{\alpha\beta}{}_{34}$, where $2!\delta^{\alpha\beta}{}_{\rho\sigma} = \varepsilon^{\alpha\beta\iota k}\varepsilon_{\rho\sigma\iota k}$ are the generalized Kroneker's deltas [9] and $\varepsilon^{\alpha\beta\iota k}$ is the Ricci tensor so defined:

$\varepsilon^{\alpha\beta\iota\kappa}$ ={0 if at least two indexes are equal or else $\pm g^{-1/2}$ if all the indexes are different, with sign + for even permutation, otherwise sign –, with $\alpha,\beta,\iota,\kappa= 0,1,2,3$}.

Thus we may write:

$$(3.3) \qquad R_{\alpha\beta\rho\sigma}\,\varepsilon^{\alpha\beta}=R_{\rho\sigma}$$

The choice on the preceding tensor is not discriminatory since we may show that any real antisymmetric 4x4 matrix may be converted into $\varepsilon^{\alpha\beta}$ through an orthogonal transformation [10]. Moreover, it is suggested by the Bianchi identity similar to (3.2) satisfied by the contracted Riemann tensor $R_{\mu\nu}$ (note that this tensor is different from the symmetric one shown in (1.1) obtained as $R_{\beta\rho}=R_{\alpha\beta\rho\sigma}g^{\alpha\sigma}$):

$$(3.4) \qquad \nabla_\sigma R_{\mu\nu}+\nabla_\mu R_{\nu\sigma}+\nabla_\nu R_{\sigma\mu}=0 \qquad (\mu=0,1,2,3)$$

Hence one could derive Einstein's gravitational equation relatively to an antisymmetric Riemann contracted tensor on applying the variational principle to the curvature scalar $R=\varepsilon^{\alpha\beta}\varepsilon^{\rho\sigma}R_{\alpha\beta\rho\sigma}$ similarly to the canonical fashion [11]. This procedure would lead to introduce an antisymmetric energy-momentum tensor not unknown in literature though without luck [12,13]. Therefore we have:

$$(3.5) \qquad R_{\mu\nu}=f(T_{\mu\nu}),$$ with f linear function,

Finally, the next step is to try to *identify* the relativistic vortex with the Riemann contracted tensor, hence obtaining:

$$(3.6a) \qquad \nabla_\sigma R_{\mu\nu}=S(\nabla_\mu k_{\nu\sigma}-\nabla_\nu k_{\mu\sigma}) \qquad (\mu,\nu,\sigma=0,1,2,3)$$

with the constant S of dimensions [length]$^{-1}$≡[angular momentum]$^{-1}$[mass] to be determined. We remind that the Riemann tensor is of dimensions [length]$^{-2}$. Someone could think of S as the inverse of Compton wavelength h/mc relatively to a mass m with h the Planck's constant. But this is but a hypothesis. Perhaps S might be a new constant. Only the experiment can establish it, but in any case it would be an index of "opacity" [14] (in analogy with the photons) directly related to the mean free path of the gravitons in matter. There would be no novelty if branches of Physics concealed themselves in other ones without any apparent connection in meaning with them like quantum mechanics with regard to the canonical transformations apart the Planck's constant [15]. Analogously with continuum if in equ. (3.6a) $k_{\mu\nu}=0$ at any point of Space-Time, so the metric is conserved, then $R_{\mu\nu}$=const, that is to say, the gravitational field does not depend on the direction and the geodesic line of a free falling frame is a right line, i.e.: $\partial_\tau^2 x^\mu=0$ as it follows at once by differentiating the integral of motion:

$$(3.7) \qquad g_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau}=c^2$$

Moreover, (3.6a) is defined at less of a coordinates transformation like Einstein's equation. I believe that equ. (3.6a) is a good candidate equation that could interpret the propagation of the deformation of metric. It just resembles the IV Maxwell's equation in its mathematical form: the Riemann tensor and the right-hand side in the second quadratic form respectively play the role of the electric and magnetic fields. One could say that if in (3.6a) there is variation of energy-momentum tensor *(source)* there will be emission of gravitational radiation *(effect)* and deformation of metric $(k_{\mu\nu}\neq 0)$ for a given proper time.

4. Simplifications and resolution definition

We may try to express (3.6a) in simpler form to be interpreted. For simplicity let us take the right-hand side:

$$(4.1)\qquad \nabla_\mu k_{\nu\sigma}-\nabla_\nu k_{\mu\sigma}$$

By reminding that:

$$\nabla_\mu k_{\nu\sigma}=\partial_\mu k_{\nu\sigma}-\Gamma^\rho{}_{\mu\nu}\,k_{\rho\sigma}-\Gamma^\rho{}_{\mu\sigma}\,k_{\nu\rho}$$
$$\nabla_\nu k_{\mu\sigma}=\partial_\nu k_{\mu\sigma}-\Gamma^\rho{}_{\nu\mu}\,k_{\rho\sigma}-\Gamma^\rho{}_{\nu\sigma}\,k_{\mu\rho}$$

with $\Gamma^\rho{}_{\mu\nu}=½g^{\kappa\rho}(\partial_\mu g_{\nu\kappa}+\partial_\nu g_{\kappa\mu}-\partial_\kappa g_{\mu\nu})$

and on inserting it into (4.1) we have:

$$(4.2)\qquad \nabla_\mu k_{\nu\sigma}-\nabla_\nu k_{\mu\sigma}=\partial_\mu k_{\nu\sigma}-\Gamma^\rho{}_{\mu\sigma}k_{\nu\rho}-\partial_\nu k_{\mu\sigma}-\Gamma^\rho{}_{\nu\sigma}k_{\mu\rho}$$

In the previous expression we have taken advantage the symmetry of Christoffel symbols versus inferior indices. Now, by recalling the definition (2.2) of $k_{\mu\nu}$ with units $c=1$ and using Schwartz theorem easily verifiable for partial derivatives we express the whole in function of the metric tensor:

$$(4.3)\qquad =½\,[\partial_\mu\partial_\tau g_{\nu\sigma}-\Gamma^\rho{}_{\mu\sigma}\partial_\tau g_{\nu\rho}-\partial_\nu\partial_\tau g_{\mu\sigma}+\Gamma^\rho{}_{\nu\sigma}\,\partial_\tau g_{\mu\rho}]$$
$$(4.4)\qquad =(1/2)\partial_\tau(\partial_\mu g_{\nu\sigma}-\partial_\nu g_{\mu\sigma})-(1/2)\Gamma^\rho{}_{\mu\sigma}\partial_\tau g_{\nu\rho}+(1/2)\Gamma^\rho{}_{\nu\sigma}\partial_\tau g_{\mu\rho}$$

From Ricci theorem according to which the covariant derivative of the metric tensor vanish, we obtain:

$$\nabla_\mu g_{\nu\sigma}=\partial_\mu g_{\nu\sigma}-\Gamma^\rho{}_{\mu\nu}g_{\rho\sigma}-\Gamma^\rho{}_{\mu\sigma}g_{\nu\rho}=0\ \Leftrightarrow$$
$$\partial_\mu g_{\nu\sigma}=\Gamma^\rho{}_{\mu\nu}\,g_{\rho\sigma}+\Gamma^\rho{}_{\mu\sigma}\,g_{\nu\rho}$$

We develop now the difference in the first term of (4.4):

$$(4.5)\qquad =½[\partial_\tau(\Gamma^\rho{}_{\mu\nu}g_{\rho\sigma}+\Gamma^\rho{}_{\mu\sigma}g_{\nu\rho}-\Gamma^\rho{}_{\nu\mu}g_{\rho\sigma}-\Gamma^\rho{}_{\nu\sigma}g_{\mu\rho})-\Gamma^\rho{}_{\mu\sigma}\partial_\tau g_{\nu\rho}+\Gamma^\rho{}_{\nu\sigma}\partial_\tau g_{\mu\rho}]$$

through the position $\Gamma^\rho{}_{\mu\sigma}\partial_\tau g_{\nu\rho}=\partial_\tau(\Gamma^\rho{}_{\mu\sigma}g_{\nu\rho})-g_{\nu\rho}\partial_\tau\Gamma^\rho{}_{\mu\sigma}$
and $\Gamma^\rho{}_{\nu\sigma}\partial_\tau g_{\mu\rho}=\partial_\tau(\Gamma^\rho{}_{\nu\sigma}g_{\mu\rho})-g_{\mu\rho}\partial_\tau\Gamma^\rho{}_{\nu\sigma}$ we have:

$$\begin{aligned}(4.6)\qquad &=½[\partial_\tau(\Gamma^\rho{}_{\mu\sigma}g_{\nu\rho})-\partial_\tau(\Gamma^\rho{}_{\nu\sigma}g_{\mu\rho})-\partial_\tau(\Gamma^\rho{}_{\mu\sigma}g_{\nu\rho})+g_{\nu\rho}\partial_\tau\Gamma^\rho{}_{\mu\sigma}+\partial_\tau(\Gamma^\rho{}_{\nu\sigma}g_{\mu\rho})-\\ &\qquad -g_{\mu\rho}\partial_\tau\Gamma^\rho{}_{\nu\sigma}]\\ &=1/2[g_{\nu\rho}\partial_\tau\Gamma^\rho{}_{\mu\sigma}-g_{\mu\rho}\partial_\tau\Gamma^\rho{}_{\nu\sigma}]\\ &=½\partial_\tau\,[\{\mu\sigma,\nu\}-\{\nu\sigma,\mu\}]\end{aligned}$$

where $\{\mu\sigma,\nu\}=½(\partial_\mu g_{\sigma\nu}+\partial_\sigma g_{\mu\nu}-\partial_\nu g_{\mu\sigma})$ are the Christoffel symbols of the first kind and finally:

$$(4.7)\qquad =½\partial_\tau[\partial_\mu g_{\nu\sigma}-\partial_\nu g_{\mu\sigma}]$$
$$(4.8)\qquad =\partial_\mu k_{\nu\sigma}-\partial_\nu k_{\mu\sigma}$$

This result is similar to the classical expression of the rotor components, but in this case the rotor is meant in a curved space at four dimensions [16,17] according to the Cisotti definition apart a multiplication factor (the covariant derivative is denoted by a slash):

$$\mathbf{R}=rot\mathbf{T},\quad R_{\sigma}{}^{\delta\gamma}=\varepsilon^{\delta\gamma\alpha\beta}T_{\sigma\beta/\alpha} \tag{4.9}$$

In fact we may easily write (3.6a) in a form evidencing the expression of the rotor (for plainness we use capital letters for $k_{\mu\nu}$) :

$$\varepsilon^{\delta\gamma\alpha\beta}\nabla_{\sigma}R_{\alpha\beta}=2S\varepsilon^{\delta\gamma\alpha\beta}K_{\sigma\beta/\alpha}, \tag{3.6 b}$$

By an ulterior rotation the expression of a typical radiation problem comes out (the double rotor of the tensor $K_{\mu\nu}$ on right-hand side of (3.6b) will be derived later):

$$rot(rot\,\mathbf{K})_{\mu\sigma}=(grad\,div\mathbf{K})_{\mu\sigma}-(/\nabla/^{2}\mathbf{K})_{\mu\sigma}-K_{\mu}{}^{\rho}\mathfrak{R}_{\rho\sigma}+K_{\beta}{}^{\alpha}R_{\sigma\alpha\tau\mu}g^{\beta\tau} \tag{4.10}$$

where $\mathfrak{R}_{\rho\nu}$ is the symmetric Riemann contracted tensor linked to the gravitational Einstein's equations (1.1). We deduce how to pass from the antisymmetric Riemann contracted tensor $R_{\mu\nu}$ to the symmetric one $\mathfrak{R}_{\mu\nu}$ in order to express both the sides of (3.6b) in function of the antisymmetric energy-momentum tensor $T_{\mu\nu}$:

$$4g^{\mu\sigma}R_{\mu\nu\rho\sigma}=\varepsilon_{\mu\nu}R_{\rho\sigma}g^{\mu\sigma}=4\mathfrak{R}_{\nu\rho}, \tag{4.11}$$

with $\varepsilon_{\mu\nu}$ inverse matrix of $\varepsilon^{\mu\nu}$.

Equ. (4.11) comes out since $4R_{\mu\nu\rho\sigma}=\varepsilon_{\mu\nu}R_{\rho\sigma}$ implies $R_{\rho\sigma}=R_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu}$.

We may conversely express $R_{\mu\nu}$ in equ. (3.5) in explicit form. Indeed, by using equ. (1.1), with $\Upsilon_{\beta\mu}$ taking place of $T_{\beta\mu}$:

$$4R_{\mu\nu}=g_{\alpha\nu}\mathfrak{R}_{\beta\mu}\,\varepsilon^{\alpha\beta}=-g_{\alpha\nu}k(\Upsilon_{\beta\mu}-1/2g_{\beta\mu}\Upsilon)\varepsilon^{\alpha\beta} \tag{4.12}$$

we have $\quad 4T_{\mu\nu}=g_{\alpha\nu}(\Upsilon_{\beta\mu}-1/2g_{\beta\mu}\Upsilon)\varepsilon^{\alpha\beta}$

Likewise, (4.12) follows since $4R_{\alpha\beta\mu\nu}=g_{\alpha\nu}\mathfrak{R}_{\beta\mu}$ implies $\mathfrak{R}_{\beta\mu}=R_{\alpha\beta\mu\nu}g^{\alpha\nu}$.

Now let us demonstrate (4.10). From the commutation relations of tensors [18] we have:

$$K_{\rho\sigma/\tau\mu}-K_{\rho\sigma/\mu\tau}=-K_{\alpha\sigma}R_{\rho}{}^{\alpha}{}_{\tau\mu}-K_{\rho\alpha}R_{\sigma}{}^{\alpha}{}_{\tau\mu}\quad\text{so that} \tag{4.13}$$

$$\begin{aligned}K_{\alpha\sigma/}{}^{\alpha}{}_{\mu}-K_{\alpha\sigma/\mu}{}^{\alpha}&=(K_{\rho\sigma/\tau\mu}-K_{\rho\sigma/\mu\tau})g^{\rho\tau}=(-K_{\alpha\sigma}R_{\rho}{}^{\alpha}{}_{\tau\mu}-K_{\rho\alpha}R_{\sigma}{}^{\alpha}{}_{\tau\mu})g^{\rho\tau}\\&=(-K^{\alpha}{}_{\sigma}R_{\rho\alpha\tau\mu}-K_{\rho}{}^{\alpha}R_{\sigma\alpha\tau\mu})g^{\rho\tau}\\&=K^{\alpha}{}_{\sigma}\mathfrak{R}_{\alpha\mu}-K_{\rho}{}^{\alpha}R_{\sigma\alpha\tau\mu}g^{\rho\tau},\ \text{where }\mathfrak{R}_{\alpha\mu}=-R_{\alpha\mu}=R_{\rho\alpha\mu\tau}g^{\rho\tau}.\end{aligned} \tag{4.14}$$

Let us now take into account the antisymmetric tensor:

$$\omega_{\mu\alpha\sigma}{}^{/\alpha}=K_{\mu\sigma/\alpha}{}^{\alpha}-K_{\alpha\sigma/\mu}{}^{\alpha}, \tag{4.15}$$

we recognize in the first term the Laplace operator.
The second term will assume the following expression via the commutation relations (4.14):

$$K_{\alpha\sigma/\mu}{}^{\alpha}=-K^{\alpha}{}_{\sigma}\mathfrak{R}_{\alpha\mu}+K_{\beta}{}^{\alpha}R_{\sigma\alpha\tau\mu}g^{\beta\tau}+K_{\alpha\sigma/}{}^{\alpha}{}_{\mu} \tag{4.16}$$

hence, on inserting (4.16) in (4.15) we have:

$$K_{\mu\sigma/\alpha}{}^{\alpha}=\omega_{\mu\alpha\sigma}{}^{/\alpha}+K_{\alpha\sigma/\mu}{}^{\alpha}\qquad\text{namely:} \tag{4.17}$$

$$(4.18)\qquad |\nabla|^2 K=\omega_{\mu\alpha\sigma}{}^{/\alpha}-K^{\alpha}{}_{\sigma}\mathfrak{R}_{\alpha\mu}+K_{\beta}{}^{\alpha}R_{\sigma\alpha\tau\mu}g^{\beta\tau}+K_{\alpha\sigma/}{}^{\alpha}{}_{\mu}.$$

We recognize in the last term the operator $grad(divK)$, while in the first one $-rot(rotK)$.

In fact:

$$-\omega_{\mu\alpha\sigma}{}^{/\alpha}=K_{\alpha\sigma/\mu}{}^{\alpha}-K_{\mu\sigma/\alpha}{}^{\alpha}=(g_{\mu}{}^{p}g_{\alpha}{}^{q}-g_{\mu}{}^{q}g_{\alpha}{}^{p})K_{q\sigma/p}{}^{\alpha}=(1/2)\varepsilon_{\mu\alpha\delta\gamma}\,\varepsilon^{\delta\gamma pq}K_{q\sigma/p}{}^{\alpha}.$$

We finally have the following equation:

$$(4.19a)\qquad (|\nabla|^2\mathbf{K})_{\mu\sigma}-(grad\ div\mathbf{K})_{\mu\sigma}=S_{\mu\sigma}\ ,\quad \text{with}$$

$$(4.19\ b)\qquad S_{\mu\sigma}=-K_{\mu}{}^{\alpha}\mathfrak{R}_{\alpha\sigma}+K_{\beta}{}^{\alpha}R_{\sigma\alpha\tau\mu}g^{\beta\tau}\ -$$

$$-\frac{1}{4S}\varepsilon_{\mu\nu\delta\gamma}\,\varepsilon^{\delta\gamma\alpha\beta}R_{\alpha\beta}{}^{/}{}_{\sigma}{}^{\nu}$$

representing the source term. Equ. (4.19a) is quite similar to Navier's equation of elasticity, so we may find a particular integral solution in complete analogy whit that [19]:

$$(4.20a)\qquad K_{\mu\sigma}=-\frac{3}{2\pi^2}\int\left\{-\frac{S_{\mu\sigma}(\xi)}{r^2}-\left(\frac{1}{r^2}\right)_{/\mu}(x^{\rho}-\xi^{\rho})S_{\rho\sigma}(\xi)\right\}d^4\xi$$

where the integral is with respect to the source coordinates ξ^{μ} and r is the cronotope distance from the field point X^{μ} to the source point ξ^{μ} wherein the energy momentum tensor is not null. In the preceding equation it is easy to uncouple the time coordinate from the spatial ones on using gaussian coordinates $g_{00}=1,\ g_{0k}=0$ with $k=1,2,3$ such that $d\tau=dt$ and $d^4\xi=d^3x'd\tau$, with x' space coordinates of the source. Moreover, (4.20a) evidently is an integral equation for the presence of $K_{\mu\sigma}$ in $S_{\mu\sigma}$. Moreover, since the solution is that of a typical problem of delayed potentials it is plain that right-hand side is to calculate at the time $\tau - r/v_g$ with v_g the velocity of waves propagation[ii] .Therefore, by carrying out (4.20) we may write:

$$(4.20b)\qquad K_{\mu\sigma}=\int N_{\sigma\alpha\tau\mu}{}^{\beta}K_{\beta}{}^{\alpha}d^4\xi+y_{\mu\sigma}$$

with

$$N_{\sigma\alpha\tau\mu}{}^{\beta}=\frac{3}{2\pi^2}\left[\left(\frac{-\delta_{\mu}{}^{\beta}R_{\alpha\sigma}+R_{\sigma\alpha}{}^{\beta}{}_{\mu}}{r^2}\right)+\left(\frac{1}{r^2}\right)_{/\mu}\cdot(x^{\rho}-\xi^{\rho})(-\delta_{\rho}{}^{\beta}R_{\alpha\sigma}+R_{\sigma\alpha}{}^{\beta}{}_{\rho})\right]$$

and

$$y_{\mu\sigma}=-\frac{3}{S4\pi^2}\int\left\{\frac{R_{\mu\nu}{}^{/}{}_{\sigma}{}^{\nu}}{r^2}+\left(\frac{1}{r^2}\right)_{/\mu}\cdot(x^{\rho}-\xi^{\rho})R_{\rho\nu}{}^{/}{}_{\sigma}{}^{\nu}\right\}d\xi^4$$

On searching for a solution of (4.20b) by successive approximations we may choose $K_{\mu\sigma}=0$ $(g_{\mu\sigma}=cost)$ as zeroth order approximation. To the first order we assume $K_{\mu\sigma}=y_{\mu\sigma}$ so that the next approximation will be:

$$(4.20c) \qquad K_{\mu\sigma} = \int N_{\sigma\alpha\tau\mu}{}^{\beta} y_{\beta}{}^{\alpha} d^4\xi + y_{\mu\sigma}$$

In order to understand how to solve the homogenous equation, for plainness we reason in rectangular coordinates with reference to the semi-infinite region $dx^0=\tau>0$; we may find the solution:

$$(4.21) \qquad K_{\mu\sigma}=\varphi_{\mu\sigma}+ \tau\psi_{\sigma/\mu}$$

where $\varphi_{\mu\sigma}$ and ψ_{σ} are harmonic functions. By substituting from (4.21) into (4.19a) and neglecting the $S_{\mu\sigma}$ term we find:

$$(4.22a) \qquad [\psi_{\sigma/0} - \varphi_{\alpha\sigma}{}^{/\alpha}]_{/\mu}=0$$

Thus, $\varphi_{\mu\sigma}$ turns out to be related to ψ_{σ}:

$$(4.22b) \qquad \psi_{\sigma/0} = \varphi_{\alpha\sigma}{}^{/\alpha}$$

Relatively to boundary conditions on $\varphi_{\mu\sigma}$, it may be shown [20] that a problem involving an equation of Navier's type is reduced to the Dirichelet problem in potential theory in the case of special form of the solution (4.21). Actually, with regard to initial conditions:

$$(4.23) \qquad K_{\mu\sigma}(0, x^1, x^2, x^3)=\varphi_{\mu\sigma}(0, x^1, x^2, x^3)= f_{\mu\sigma}(x^1, x^2, x^3)$$

and on forming its Fourier integral (by integrating each variable over the range $(-\infty,+\infty)$):

$$(4.24) \qquad g_{\mu\sigma}(\alpha,\beta,\gamma)=\frac{1}{2\pi^3}\int f_{\mu\sigma}(\varsigma,\eta,\chi)\, e^{i(\alpha\varsigma+\beta\eta+\gamma\chi)}\, d\varsigma\, d\eta\, d\chi$$

we obtain the following integral solution:

$$(4.25) \qquad K_{0\sigma}=\int\left\{ g_{0\sigma}(\alpha,\beta,\gamma)+\tau\left[\delta g_{0\sigma}+i\left(\alpha g_{1\sigma}+\beta g_{2\sigma}+\gamma g_{3\sigma}\right)\right]\right\} \times e^{\delta\tau+i\left(\alpha x^1+\beta x^2+\gamma x^3\right)} d\alpha\, d\beta\, d\gamma$$

$$K_{1\sigma}=\int\left\{ g_{1\sigma}(\alpha,\beta,\gamma)+i\left(\frac{\alpha\tau}{\delta}\right)\left[\delta g_{1\sigma}+i\left(\alpha g_{1\sigma}+\beta g_{2\sigma}+\gamma g_{3\sigma}\right)\right]\right\} \times e^{\delta\tau+i\left(\alpha x^1+\beta x^2+\gamma x^3\right)} d\alpha\, d\beta\, d\gamma$$

… similarly for the other components.

Here $\delta= -(\alpha^2+\beta^2+\gamma^2)^{1/2}$ and i the imaginary constant.

We find that the solutions (4.25) are reduced to the initial condition (4.23) as $\tau=0$ and asymptotically vanish as τ tends to ∞, like typical solutions in propagation of heat.

5. Lagrangian formulation

A useful expression of (3.6) may simply be obtained by means of Eulero-Lagrange formulation if we compare (4.19a) in free field $(S_{\mu\sigma}=0)$ with the analogous one in electromagnetism; we reason in rectangular coordinates at first generalizing the results afterwards on substituting covariant derivatives for ordinaries ones. Therefore, reminding the well-known electromagnetic relations:

$$(5.1) \qquad \partial^{\mu}\partial_{\mu}A^{\nu}-\partial^{\nu}\partial_{\mu}A^{\mu}=|\nabla|^{2}A^{\nu}-(\mathrm{grad\ div}\ \mathbf{A})=\partial_{\mu}F^{\mu\nu}=0,$$

where A^{μ} is the 4-potential and $F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}$
and the related lagrangian :

$$(5.2) \qquad L=-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$$

Equation (4.19a) analogously comes out from the following lagrangian:

$$(5.3) \qquad L=\Phi_{\nu\sigma\mu}\Phi^{\nu\sigma\mu}, \quad \text{where} \quad \Phi_{\nu\sigma\mu}=K_{\nu\sigma/\mu}-K_{\mu\sigma/\nu}$$

Similarly the expression of the energy-momentum tensor of free field will be:

$$(5.4) \qquad t_{\mu}{}^{\nu}=K^{\alpha\sigma}{}_{/\mu}\frac{\partial L}{\partial K^{\alpha\sigma}{}_{/\nu}}-\delta_{\mu}{}^{\nu}L$$

namely, by some manipulations:

$$(5.5) \qquad t_{\mu}{}^{\nu}=4\Phi_{\mu}{}^{\sigma\alpha}\Phi^{\nu}{}_{\sigma\alpha}-\delta_{\mu}{}^{\nu}\Phi_{\alpha\sigma\rho}\Phi^{\alpha\sigma\rho}$$

so that the flow of energy in radial direction from a system of masses will be, as well-known:

$$(5.6) \qquad \frac{d^{2}E}{dt\,r^{2}d\Omega}=t_{s}{}^{0}n^{s}$$

where n^{s} is the versor in radial direction and $d\Omega$ the angular differential.

Conclusions

The aim of this paper has been that of trying to lay the groundwork for a theory of geometrodynamics, which, if right, could be of great moment in matter of generation of gravitational waves. Moreover, simulations of gravitational energy losses should fit experimental results interpreted so far on using Einstein's equation alone. It would permit us to know the constant value S of equ. (3.6). Further, in my opinion gravitational radiation should be completely revised in its framework and surprises would not be excluded in its detection, as for instance waves would propagate in phase according to (4.25) relatively to symmetric initial conditions and not in counter-phase as believed, thus making inefficacious interferometric detection.

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[i] Really, if we try to contract the Riemann tensor by means of $g^{\alpha\beta}$ ie $R_{\alpha\beta\rho\sigma}g^{\alpha\beta}=R_{\rho\sigma}$, the result evidently is null in account of the antisymmetric property of the Riemann tensor versus the first or second couple of indexes.

[ii] By evaluating the characteristic varieties of (4.19a) according to [2] the velocity of waves propagation turns out to be the velocity of light ***C***, as usual.